On Gated Gaussian Impulsive Noise in M-QAM with Optimum Receivers

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Abstract—This paper presents an analysis of the gated Gaussian impulsive noise and its effects on M-ary Quadrature Amplitude Modulation (M-QAM) schemes. In the approach, both amplitude variation and noisy pulse duration can be characterized as a modulation of the impulsive noise component by a discrete (binary or m-ary) random process. New exact expressions are presented for the probability density function, autocorrelation function and power spectral density of the noise, as well as for the bit error probability of M-QAM, considering the maximum a posteriori probability optimum receiver. An important aspect of the proposed approach is the fact that the discrete random process incorporates the main parameters of the impulsive noise, such as amplitude, duration, instants in which the noise is added and time intervals between instants in which the noise is added.

Index Terms—Gated impulsive noise, double gated impulsive noise, digital modulation, bit error probability.

I. INTRODUCTION

GATED Gaussian noise or impulsive interference terminology has been used since 2001, when the Digital Television Group, leaded by BBC R&D with the participation of Sony, Philips, Rohde & Schwarz, Zarlink and STMicroelectronics, was created. The purpose of the group was to establish a set of waveforms for tests, and methods to be used to adequately represent the effect of impulsive interference in digital terrestrial television (DTT). A resume of the results of the studies of the group was presented in [1], concerning the contribution of the authors to the group of studies of BBC, from October 2001 to October 2002. Their contribution consisted in the acquisition and statistical analysis of the real impulsive interference for the proposal of a model of noise and conducting measurements in laboratory for validation and simplification of the proposed model.

According to [1], the interference caused by equipment, such as appliances, central heating thermostats, operating switches, rectifiers and ignition systems, occurs in bursts whose duration and intervals between occurrences may be characterized by parameters obtained from measurements. Those bursts, when occur, present amplitude variations that are statistically characterized by a Gaussian distribution with zero mean and a given variance.

The waveforms presented in [1] can be seen as sample functions of a continuous-time random process, resulting from the modulation by an m-ary continuous-time process represented by C(t). The mathematical approach that characterizes the impulsive interference is the main contribution of this research. The advantage that arises from representing the burst behaviour of the impulsive interference as a product of a discrete process C(t) by a white Gaussian process η(t), with zero mean and variance σ^2, is that some burst aspects, such as duration and time intervals in which they are absent, may be characterized by C(t).

The evaluation of the influence of the impulsive interference on the performance of optimum receivers for M-QAM modulation schemes is also discussed. This evaluation can be performed after the determination of the probability density function of the composed noise η(t) = η(t) + C(t)η(t), in which η(t) represents a white Gaussian process, with zero mean and variance σ^2. This is obtained by expressing the probability density function of the discrete process C(t) in terms of an impulse. Therefore, such model is more general than the one presented in [2], which considers C(t) a Bernoulli process.

In addition to the characterization of the impulsive noise, the paper also addresses its influence on the performance of the maximum a posteriori probability (MAP) receiver for the digital rectangular M-QAM (M-ary Quadrature Amplitude Modulation).

Impulsive noise is present in many digital communication systems [3]–[6], and also in Power Line Communication (PLC) networks [7], [8], considering equipment subject to partial discharges [9], [10]. The mitigation of the effects of impulsive noise is important in digital television [11] and OFDM modulation systems [12], [13], wireless communications [14], [15] and development of blind techniques for signal processing in impulsive channels [16].

Concerning digital television channels, GAWGN (Gated Additive White Gaussian Noise) and G2AWGN (Double Gated Additive White Gaussian Noise) are widely used models, because they are easy to implement and analyze, and they strongly agree with the corresponding experimental measurements [19], [20].

The model was used by the digital television research group at BBC (British Broadcasting Corporation), and allows industry and telecommunications companies to perform impulsive noise simulation. Using those techniques, the development of techniques for mitigating the adverse effects of impulsive noise...
II. MATHEMATICAL CHARACTERIZATION OF THE PROPOSED MODEL

An approach for the mathematical time domain characterization of impulsive noise models GA WGN and $G^2$AWGN is presented in this section. The proposed approach uses a digital random modulating signal $C(t) = C_1(t)C_2(t)$ for amplitude modulation of a white Gaussian noise $\eta(t)$, with zero mean and variance $\sigma^2_\eta$, for characterizing parameters such as amplitude, duration, instants in which the noise is added and time intervals between instants in which the impulsive noise $C(t)\eta(t)$ is added. The random process $C_1(t)$ represents the occurrence of bursts and $C_2(t)$ represents the occurrence of pulses. The pulses occur in the interval of duration of the bursts.

The modulating signal $C(t)$ is a binary digital random process, which assumes values in the set $\{0, 1\}$, or $m$-ary digital random process, which assumes values in the set $\{c_0, c_2, \ldots, c_{m-1}\}$, according to a mass probability function. The total noise in the system is the sum of the impulsive component $C(t)\eta(t)$ and the permanent noise component represented by a white Gaussian process $\eta_0(t)$ of zero mean and variance $\sigma^2_\eta$. An important aspect of using a modulating signal $C(t)$ is the fact that parameters of the impulsive noise, such as amplitude, duration, instants in which the noise is added and time intervals between occurrences of pulses and bursts can be modeled by that random process.

In the present work, an additive noise channel is considered in which the received signal $r(t)$ can be written as

$$r(t) = s(t) + \eta(t),$$

where $s(t)$ represents the transmitted signal and

$$\eta(t) = C_1(t)C_2(t)\eta_1(t) + \eta_0(t)$$

represents the noise composed by an impulsive noisy component $C_1(t)C_2(t)\eta_1(t)$ and by the permanent noisy component $\eta_0(t)$. According to the values assumed by $C_1(t)$ and $C_2(t)$, four different models can be proposed for the noise. In the first model, $C_1(t) = 1$ and $C_2(t)$ randomly assumes values in the set $\{0, 1\}$. This model is referred to as Gated Binary Gaussian Impulsive Noise. In the second model, $C_1(t)$ and $C_2(t)$ randomly assume values in the set $\{0, 1, \ldots, m\}$. This model is referred to as Double Gated Binary Gaussian Impulsive Noise. In the third model, $C_1(t) = 1$ and $C_2(t)$ randomly assumes values in the discrete set $\{c_0, c_2, \ldots, c_{m-1}\}$. This model is referred to as Gated Multilevel Gaussian Impulsive Noise. In the last model, $C_1(t)$ and $C_2(t)$ randomly assume values in the discrete sets $\{c_0, c_1, \ldots, c_{m-1}\}$ and $\{c_0, c_2, \ldots, c_{m-1}\}$. This model is referred to as Double Gated Multilevel Gaussian Impulsive Noise.

III. GATED BINARY GAUSSIAN IMPULSIVE NOISE

In this model, the impulsive component is obtained multiplying the process $\eta_1(t)$ by a square wave $C_2(t)$. The time interval in which this square wave assumes unitary value with probability $p_2$ is $\alpha T_2$, with $0 \leq \alpha \leq 1$. The obtained modulated signal is a sequence of samples of an AWGN process $\eta_1(t)$ added to the process which models the permanent noise,
\[ \eta(t) = \eta_0(t) + C_2(t) \eta(t), \quad (3) \]
in which the modulating signal \( C_2(t) \) randomly assumes amplitudes \( m_l \) in the set \( \{0, 1\} \) and can be represented by a sequence of pulses
\[ C_2(t) = \sum_{l=-\infty}^{\infty} m_l P_{R_2}(t - lT_2), \quad (4) \]
in which \( m_l \) is a sequence of i.i.d. r.v. with probability distribution \( P\{m_l = 1\} = p_2 \) and \( P\{m_l = 0\} = 1 - p_2 \). Additionally, the square pulse \( P_{R_2}(t) \) of duration \( T_2 \) is defined as
\[ P_{R_2}(t) = \begin{cases} 1, & 0 \leq t \leq \alpha T_2 \\ 0, & \text{otherwise.} \end{cases} \quad (5) \]
Thus, the total impulsive noise GAWGN can be expressed as
\[ \eta(t) = \eta_0(t) + \sum_{l=-\infty}^{\infty} m_l P_{R_2}(t - lT_2) \eta(t). \quad (6) \]

Figure 1 presents a realization of that random process obtained from Equation 3.

**A. Probability Density Function**

For obtaining the probability density function (pdf) of the noise in Equation 3, one uses the pdf of the random variable (r.v.) \( Y(t) = C_2(t) \eta(t) \). Based on the fact that the pdf of a sum of two independent r.v. is given by the convolution of the individual pdf’s [29], the pdf of \( \eta(t) \) is obtained by the convolution of the pdf of \( \eta_0(t) \) with the pdf of \( Y(t) \). Thus,
\[ f_{\eta(t)}(\eta) = f_{\eta_0(t)}(\eta) \ast f_Y(\eta), \quad (7) \]
and
\[ f_{\eta(t)}(\eta) = f_{\eta_0(t)}(\eta) \ast \left[ (1 - p_2) \delta(\eta) + p_2 f_{\eta(t)}(\eta) \right] \]
\[ = f_{\eta_0(t)}(\eta) \ast \left[ (1 - p_2) \delta(\eta) + p_2 f_{\eta(t)}(\eta) \right] \]
\[ = (1 - p_2) f_{\eta_0(t)}(\eta) + p_2 f_{\eta(t)}(\eta) \ast f_{\eta(t)}(\eta). \quad (8) \]

Substituting the pdf
\[ f_{\eta_0(t)}(\eta) = \frac{1}{\sqrt{2\pi \sigma^2_\eta}} \exp\left(\frac{-\eta^2}{2\sigma^2_\eta}\right) \quad (9) \]
and
\[ f_{\eta(t)}(\eta) = \frac{1}{\sqrt{2\pi \sigma^2_\eta}} \exp\left(\frac{-\eta^2}{2\sigma^2_\eta}\right) \quad (10) \]
in Equation 8 and after some algebraic manipulations, one can write \( f_{\eta(t)}(\eta) \) in the form
\[ f_{\eta(t)}(\eta) = \alpha_2 \left[ \frac{1}{2\pi(\sigma^2_\eta + \sigma^2_\eta)} \exp\left(\frac{-\eta^2}{2(\sigma^2_\eta + \sigma^2_\eta)}\right) \right] \]
\[ + (1 - \alpha_2) \left[ \frac{1}{\sqrt{2\pi \sigma^2_\eta}} \exp\left(\frac{-\eta^2}{2\sigma^2_\eta}\right) \right], \quad (11) \]
Assuming that \( C_2(t) = 1 \) with probability \( p_2 \) in intervals of duration \( \alpha T_2 \) and \( C_2(t) = 0 \) with probability \( (1 - p_2) \) in intervals of duration \( (1 - \alpha) T_2 \), the pdf of \( \eta(t) \) can be written as
\[ f_{\eta(t)}(\eta) = \alpha_2 \left[ \frac{1}{2\pi(\sigma^2_\eta + \sigma^2_\eta)} \exp\left(\frac{-\eta^2}{2(\sigma^2_\eta + \sigma^2_\eta)}\right) \right] \]
\[ + (1 - \alpha_2) \left[ \frac{1}{\sqrt{2\pi \sigma^2_\eta}} \exp\left(\frac{-\eta^2}{2\sigma^2_\eta}\right) \right], \quad (12) \]

**B. Autocorrelation Function and Power Spectral Density**

The autocorrelation function of the proposed model \( \eta(t) \) can be obtained from the definition [30]
\[ R_\eta(t, \tau) = E[\eta(t)\eta(t + \tau)] \]
\[ = E[\eta(t) + C_2(t) \eta(t)](\eta(t + \tau) + C_2(t + \tau) \eta(t + \tau)) ] \quad (13) \]

Considering that the processes \( \eta_0(t) \) and \( \eta(t) \) are zero-mean and independent, one can write \( R_\eta(t, \tau) \) as
\[ R_\eta(t, \tau) = R_{\eta_0}(\tau) + E[C_2(t)C_2(t + \tau)] R_{\eta(t)}, \quad (14) \]
in which
\[ E[C_2(t)C_2(t + \tau)] = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left[ E[\eta_0(t)\eta_0(t + \tau)] \right. \]
\[ \times P_{R_2}(t - lT_2) P_{R_2}(t + \tau + kT_2) \]
\[ = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left[ E[\eta_0(t)\eta_0(t + \tau)] \right. \]
\[ \times P_{R_2}(t - lT_2) P_{R_2}(t + \tau + kT_2) \]
and \( m_i, i \in \mathbb{Z} \), are independent and identically Bernoulli random variables with probability distribution \( P\{m_i = 1\} = p_2 \) and \( P\{m_i = 0\} = 1 - p_2 \). Therefore 15 can be written as

\[
E[C_2(t)C_2(t + \tau)] = \\
\sum_{l=\infty}^{\infty} \sum_{k=\infty}^{\infty} \sum_{k \neq l}^{\infty} E[m_1m_k]P_{R_2}(t - lT_2)P_{R_2}(t + \tau - kT_2) \\
+ \sum_{l=\infty}^{\infty} \sum_{k=-\infty}^{\infty} E[m_1^2]P_{R_2}(t - lT_2)P_{R_2}(t + \tau - lT_2) \\
= p_2^2 s_2(t) s_2(t + \tau) + p_2 \sum_{l=\infty}^{\infty} \sum_{k=-\infty}^{\infty} P_{R_2}(t - lT_2)P_{R_2}(t + \tau - lT_2),
\]

in which

\[
s_2(t) = \sum_{k=-\infty}^{\infty} P_{R_2}(t - kT_2).
\]

Applying the result presented in Equation 16 into Equation 14, one can write

\[
R_\eta(t, \tau) = R_{\eta_2}(\tau) + \\
+ p_2^2 \sum_{l=\infty}^{\infty} \sum_{k=-\infty}^{\infty} P_{R_2}(t - lT_2)P_{R_2}(t + \tau - kT_2) \frac{N_i}{2} \delta(\tau) \\
+ p_2 \sum_{l=\infty}^{\infty} P_{R_2}(t - lT_2)P_{R_2}(t + \tau - lT_2) \frac{N_i}{2} \delta(\tau).
\]

Applying the Fourier transform to \( R_\eta(t, \tau) \), regarding \( \tau \), one can write the power spectral density (PSD) as

\[
S_\eta(t, \omega) = S_{\eta_2}(\omega) + \frac{N_i p_2^2}{2} \sum_{l=\infty}^{\infty} \sum_{k=-\infty}^{\infty} P_{R_2}(t - lT_2) \\
\times \sum_{k=\infty}^{\infty} \int_{-\infty}^{\infty} P_{R_2}(t + \tau - kT_2) \delta(\tau) e^{-j\omega \tau} d\tau \\
+ \frac{p_2 N_i}{2} \sum_{l=\infty}^{\infty} P_{R_2}(t - lT_2) \\
\times \int_{-\infty}^{\infty} P_{R_2}(t + \tau - lT_2) \delta(\tau) e^{-j\omega \tau} d\tau.
\]

Applying the sampling property of the Dirac’s impulse, one can write the PSD as

\[
S_\eta(t, \omega) = S_{\eta_2}(\omega) + \frac{N_i p_2^2}{2} \sum_{l=\infty}^{\infty} \sum_{k=-\infty}^{\infty} P_{R_2}(t - lT_2)P_{R_2}(t - kT_2) \\
+ \frac{p_2 N_i}{2} \sum_{l=\infty}^{\infty} P_{R_2}(t - lT_2).
\]

The double sum on Equation 20 is null because the train of rectangular pulses are orthogonal when \( k \neq l \). For eliminating the dependence of \( S_\eta(t, \omega) \) with respect to the time \( t \), one can take the temporal mean, so

\[
S_\eta(\omega) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} S_\eta(t, \omega) dt = \frac{N_0}{2} + p_2^2 \frac{N_i}{2}.
\]

IV. DOUBLE GATED BINARY GAUSSIAN IMPULSIVE NOISE

In this model, the process AWGN \( \eta_2(t) \), of zero mean and variance \( \sigma_2^2 \), is added to the process AWGN \( \eta_1(t) \), of zero mean and variance \( \sigma_1^2 \), and multiplied by two modulating binary signals, \( C_1(t) \) and \( C_2(t) \), which are formed by rectangular pulses of duration \( T_1 \) and \( T_2 \), respectively. The time interval in which \( C_1(t) \) assumes unitary value is \( \beta T_1 \) and the time interval in which \( C_2(t) \) assumes unitary value is \( \alpha T_2 \).

Considering \( b \) pulses in each burst, \( T_1 = b T_2 \). The modulated noise obtained, \( \eta(t)C_1(t)C_2(t) \), forms a sequence of noisy bursts with duration \( \beta T_1 \), in each pulse has duration \( \alpha T_2 \). The modulating signals \( C_1(t) \) and \( C_2(t) \) randomly assume the values \( m_k \) and \( m_l \), respectively, in the set \( \{0, 1\} \).

Thus, the model which describes the behavior of that noise is given by

\[
\eta(t) = \eta_2(t) + C_1(t)C_2(t) \eta_1(t),
\]

in which the modulating signal \( C_2(t) \), which is associated with the presence or absence of bursts, is represented by the sequence of rectangular pulses

\[
C_1(t) = \sum_{k=-\infty}^{\infty} m_k P_{R_1}(t - kT_1),
\]

in which the probability distribution of \( m_k \) is given by \( P\{m_k = 1\} = p_1 \) and \( P\{m_k = 0\} = 1 - p_1 \).

The modulating signal \( C_2(t) \), which is associated to the presence or absence of pulses, is represented by the sequence of rectangular pulses

\[
C_2(t) = \sum_{l=-\infty}^{\infty} m_l P_{R_2}(t - lT_2),
\]

in which the probability distribution of \( m_l \) is given by \( P\{m_l = 1\} = p_2 \) and \( P\{m_l = 0\} = 1 - p_2 \).

In both modulating signals, the rectangular pulses \( P_{R_1}(t) \) and \( P_{R_2}(t) \), of duration \( T_1 \) and \( T_2 \), are given by

\[
P_{R_1}(t) = \begin{cases} 
1, & 0 \leq t \leq \beta T_1 \\
0, & \text{otherwise}
\end{cases}
\]

in which \( 0 \leq \beta \leq 1 \), and by

\[
P_{R_2}(t) = \begin{cases} 
1, & 0 \leq t \leq \alpha T_2 \\
0, & \text{otherwise}
\end{cases}
\]

in which \( 0 \leq \alpha \leq 1 \).

Thus, the total noise can be given by

\[
\eta(t) = \eta_2(t) + C_1(t)C_2(t) \eta_1(t) = \eta_2(t) + \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} m_k m_l P_{R_1}(t - kT_1)P_{R_2}(t - lT_2) \eta_1(t).
\]
A. Probability Density Function

The procedure for determining the pdf of the process η(t) which represents the double gated binary Gaussian impulsive noise is similar to the one used in Section III-A. In this second model, both $C_1(t)$ and $C_2(t)$ switch and assume random values in the set \{0, 1\}. Hence, the joint probability density function of $C_1(t)$ and $C_2(t)$ can be written as

$$f_{C_1(t), C_2(t)}(c_1, c_2) = \sum_{k=0}^{1} \sum_{l=0}^{1} p_{1k} p_{2l} \delta(c - c_{1k} c_{2l}),$$

in which the mass probability functions (mpf) of the values assumed by $C_1(t)$ and $C_2(t)$ in the sets \{c_{10}, c_{11}, \ldots, c_{1(m_1-1)}\} and \{c_{20}, c_{21}, \ldots, c_{2(m_2-1)}\} are given by $P(C_1(t) = c_{1k}) = p_{1k}$ and $P(C_2(t) = c_{2l}) = p_{2l}$.

Thus, the expression of the total noise $\eta(t)$ can be written as

$$\eta(t) = \eta_0(t) + C_1(t)C_2(t)\eta_1(t).$$

Then, the pdf of $\eta(t)$ can be written as

$$f_{\eta(t)}(\eta) = \sum_{k=0}^{1} \sum_{l=0}^{1} \frac{p_{1k} p_{2l}}{\sqrt{2\pi(\sigma_0^2 + \sigma_k^2 c_{1k}^2 c_{2l}^2)}} \times \exp\left(-\frac{\eta^2}{2(\sigma_0^2 + \sigma_k^2 c_{1k}^2 c_{2l}^2)}\right).$$

Equation 30, as

$$f_{\eta(t)}(\eta) = \frac{\alpha\beta p_1 p_2}{\sqrt{2\pi(\sigma_0^2 + \sigma_k^2)}} \exp\left(-\frac{\eta^2}{2(\sigma_0^2 + \sigma_k^2)}\right) + \frac{(1 - \alpha\beta p_1 p_2)}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{\eta^2}{2\sigma_0^2}\right).$$

B. Autocorrelation Function and Power Spectral Density

The autocorrelation function is obtained from $\eta(t)$ given in Equation 22 and can be expressed as

$$R_\eta(t, \tau) = E[\eta(t)\eta(t + \tau)].$$

Considering that the processes $\eta_0(t)$ and $\eta_1(t)$ are independent and the probability distributions for $m_k$ and $m_\tau$, of the signals $C_1(t)$ and $C_2(t)$ given in Equations 23 and 24, one can write $R_\eta(t, \tau)$ as

$$R_\eta(t, \tau) = \frac{N_0}{2} \delta(\tau) + s_{12}(t)p_{12}\sigma_0^2 N_i.\frac{p_{12}}{2} \delta(\tau),$$

in which

$$s_{12}(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} P_{R_1}(t - kT_1)P_{R_2}(t - lT_2).$$

For eliminating the dependence of the autocorrelation with respect to $t$, one can obtain the time mean of $R_\eta(t, \tau)$, as

$$R_\eta(\tau) = \lim_{T_1 \to \infty} \frac{1}{T_1} \int_{-T_1/2}^{T_1/2} R_\eta(t, \tau)dt$$

$$= \frac{N_0}{2} \delta(\tau) + p_{12}\alpha \beta \frac{N_i}{2} \delta(\tau).$$

The PSD of $\eta(t)$ is obtained by the Fourier transform of the autocorrelation function $R_\eta(\tau)$, that is

$$S_\eta(\omega) = \int_{-\infty}^{\infty} R_\eta(\tau) \exp(-j\omega \tau) d\tau$$

$$= \frac{N_0}{2} + \alpha \beta p_1 p_2 N_i \frac{p_{12}}{2}.$$
The constants \( \eta \) is the one used to obtain the pdf of the first two models. In this model, the modulating signals \( \eta(t) \) and \( \eta(t) \) are independent, the pdf of \( \eta(t) \) can be written as

\[
f_{\eta(t)}(\eta) = \sum_{l=0}^{m_2-1} p_{2l} \frac{1}{\sqrt{2\pi(\sigma_g^2 + \sigma^2 c_{2l}^2)}} \exp \left( -\frac{\eta^2}{2(\sigma_g^2 + \sigma^2 c_{2l}^2)} \right). \tag{39}
\]

One can observe, from Equation 39, that the pdf of \( \eta(t) \) depends on values assumed by the modulating signal at instant \( t \). If the process \( \eta(t) \) assumes only two values, like random binary signals, then that sum only has two terms. Other important aspect of \( \eta(t) \) is that its pdf remains with the format of a symmetrical Gaussian pdf, since both \( \eta(t) \) and \( \eta(t) \) are zero mean.

The pdf of \( \eta(t) \) can also be written as

\[
f_{\eta(t)}(\eta) = p_{20} f_{c_{20}}(\eta) + \cdots + p_{2(m_2-1)} f_{c_{2(m_2-1)}}(\eta), \tag{40}
\]

in which

\[
f_{c_{2l}}(\eta) = \frac{1}{\sqrt{2\pi(\sigma_g^2 + \sigma^2 c_{2l}^2)}} \exp \left( -\frac{\eta^2}{2(\sigma_g^2 + \sigma^2 c_{2l}^2)} \right). \tag{41}
\]

Equation 40 is a Gaussian mixture formed by probability density functions of zero mean and variance \( \sigma_g^2 + \sigma^2 c_{2l}^2 \). The constants \( c_{2l} \) correspond to the values the signal \( C_2(t) \) can assume. In Figure 3, for instance, \( C_2(t) \) assume three equiprobable values, \( c_{20} = 0, c_{21} = 1 \) and \( c_{22} = 2 \).

Considering the probability distributions \( p_{20}, p_{21}, \ldots, p_{2(m_2-1)} \), respectively, of the levels \( c_{20}, c_{21}, \ldots, c_{2(m_2-1)} \) of the multilevel modulating signal \( C_2(t) \) and the duration \( \alpha T_2 \) of the rectangular pulses \( P_{R_2}(t) \) present in \( C_2(t) \), one can write the pdf of \( \eta(t) \), given in Equation 39, as

\[
f_{\eta(t)}(\eta) = \alpha \sum_{l=0}^{m_2-1} p_{2l} f_{c_{2l}}(\eta) + (1 - \alpha) f_{\eta(t)}(\eta), \tag{42}
\]

in which \( p_{2l} = P\{C_2(t) = c_{2l}\} \).

\[\text{B. Autocorrelation Function and Power Spectral Density}\]

The autocorrelation function is obtained from \( \eta(t) \), in Equation 37, and can be written as

\[
R_{\eta}(t, \tau) = R_{\eta_1}(t) + R_{\eta_2}(t) s_2(t) \sum_{l=0}^{m_2-1} p_{2l} c_{2l}^2
\]

\[
= \frac{N_0}{2} \delta(t) + \frac{N_i}{2} \delta(t) s_2(t) \sum_{l=0}^{m_2-1} p_{2l} c_{2l}^2,
\tag{43}
\]

in which \( s_2(t) = \sum_{l=-\infty}^{\infty} P R_2(t - l T_2) \). For eliminating the dependence of the autocorrelation function with respect to time \( t \), one can obtain the time mean of \( R_{\eta}(t, \tau) \), as

\[
R_{\eta}(\tau) = \alpha \frac{N_0}{2} \delta(\tau) + \alpha \frac{N_i}{2} \delta(\tau) \sum_{l=0}^{m_2-1} p_{2l} c_{2l}^2.
\tag{44}
\]

Then, the power spectral density can be written as

\[
S_{\xi}(\omega) = \alpha \frac{N_0}{2} + \alpha \frac{N_i}{2} \sum_{l=0}^{m_2-1} p_{2l} c_{2l}^2.
\tag{45}
\]

\[\text{VI. DOUBLE GATED MULTILEVEL GAUSSIAN IMPULSIVE NOISE}\]

In this model, the modulating signals \( C_1(t) \) and \( C_2(t) \) can assume non-negative discrete values in the sets \( \{c_{10}, c_{11}, \ldots, c_{1(m_1-1)}\} \) and \( \{c_{20}, c_{21}, \ldots, c_{2(m_2-1)}\} \). The random modulating signal \( C(t) = C_1(t) C_2(t) \) is used to randomly change the power of the impulsive noise \( \eta(t) \) added to the permanent Gaussian noise.

The model that describes the behavior of such noise is expressed as

\[
\eta(t) = \eta(t) + C_1(t) C_2(t) \eta(t),
\tag{46}
\]

in which the signals \( C_1(t) \) and \( C_2(t) \) are given respectively by

\[
C_1(t) = \sum_{k=-\infty}^{\infty} m_{1k} P R_1(t - k T_1)
\tag{47}
\]

and

\[
C_2(t) = \sum_{l=-\infty}^{\infty} m_{2l} P R_2(t - l T_2).
\tag{48}
\]

The joint pdf of \( C_1(t) \) and \( C_2(t) \) can be written as

\[
f_{C_1(t), C_2(t)}(c_1, c_2) = \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} p_{1k} p_{2l} \delta(c - c_{1k} c_{2l}).
\tag{49}
\]

Figure 4 presents a realization of the process \( \eta(t) \) obtained from Equation 46. One can observe the random behavior of the amplitudes and the duration of the bursts.
Considering the probability distributions $p$ of the process that models the double gated multilevel Gaussian impulsive noise as

$$f_{\eta(t)}(\eta) = \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} \frac{p_{1k}p_{2l}}{\sqrt{2\pi(\sigma_\eta^2 + \sigma_\epsilon^2c_1^2c_2^2)}} \times \exp\left(-\frac{\eta^2}{2(\sigma_\eta^2 + \sigma_\epsilon^2c_1^2c_2^2)}\right).$$

(50)

Considering the probability distributions $p_{10}, p_{11}, \ldots, p_{1(m_1-1)}$ and $p_{20}, p_{21}, \ldots, p_{2(m_2-1)}$ respectively of the levels $c_{10}, c_{11}, \ldots, c_{1(m_1-1)}$ and $c_{20}, c_{21}, \ldots, c_{2(m_2-1)}$, as well as the durations $\beta T_1$ and $\alpha T_2$ of the rectangular pulses $P_{R_1}(t)$ and $P_{R_2}(t)$ present in $C_1(t)$ and $C_2(t)$, the pdf $\eta(t)$ can be written as

$$f_{\eta(t)}(\eta) = \alpha^2 \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} p_{1k}p_{2l}f_{kl}(\eta) + (1 - \alpha^2)f_{\eta_1}(\eta),$$

(51)

in which

$$f_{kl}(\eta) = \frac{1}{\sqrt{2\pi(\sigma_\eta^2 + \sigma_\epsilon^2c_1^2c_2^2)}} \exp\left(-\frac{\eta^2}{2(\sigma_\eta^2 + \sigma_\epsilon^2c_1^2c_2^2)}\right).$$

(52)

### A. Probability Density Function

Following a procedure similar to the one used in previous sections concerning the other models of noise, one can write the pdf of the process that models the double gated multilevel Gaussian impulsive noise as

$$R_\eta(t, \tau) = \frac{N_0}{2} \delta(\tau) + \frac{N_2}{2} \delta(\tau)s(t) \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} p_{1k}p_{2l}c_1^2c_2^2,$$

(53)

in which $s_{12}(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} P_{R_1}(t-kT_1)P_{R_2}(t-lT_2)$. For eliminating the dependence of the autocorrelation function with respect to time $t$, one can calculate the time mean of $R_\eta(t, \tau)$ as

$$R_\eta(\tau) = \frac{N_2}{2} \delta(\tau) + \frac{N_2}{2} \delta(\tau) \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} p_{1k}p_{2l}c_1^2c_2^2.$$

(54)

The power spectral density can then be written as

$$S_\eta(\omega) = \frac{N_0}{2} + \frac{N_2}{2} \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} p_{1k}p_{2l}c_1^2c_2^2.$$

(55)

### B. Autocorrelation Function and Power Spectral Density

The autocorrelation function of the process that models the double gated multilevel Gaussian impulsive noise, given in Equation 46, can be written as

$$R_\eta(t, \tau) = \frac{N_0}{2} \delta(\tau) + \frac{N_2}{2} \delta(\tau)s(t) \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} p_{1k}p_{2l}c_1^2c_2^2,$$

(53)

VII. PERFORMANCE EVALUATION OF MAP RECEIVER WITH MODULATION M-QAM UNDER IMPULSIVE NOISE

In this section, four models of impulsive noise, presented in Sections III, IV, V and VI, are considered. A performance evaluation of MAP receiver [31] under impulsive noise is carried out. For the evaluation, square $M$-QAM modulation is considered and the bit error probability (BEP) is derived from the expressions introduced in [32].

According to Cho and Yoon [32], the BEP of $M$-QAM, given a signal-to-noise ratio $\gamma = E_b/N_0$, can be written, in terms of the constellation order, $M$, as

$$P_M(e|\gamma) = \frac{1}{\log_2 \sqrt{M}} \sum_{k=1}^{\infty} P_b(k),$$

(56)

in which $P_b(k)$ can be written as

$$P_b(k) = \frac{1}{\sqrt{M}} \sum_{i=0}^{(1-2^{-k})\sqrt{M} - 1} \left\{ w(i, k, M) \times \text{erfc}\left((2i + 1)\sqrt{3\log_2 M\gamma / 2(M - 1)}\right) \right\},$$

(57)

the coefficients $w(i, k, M)$ are given by

$$w(i, k, M) = (-1)^{\left\lfloor \frac{2^{k-1} - i}{2k-1} \right\rfloor} \left( 2^{k-1} - \left\lfloor \frac{i + 2^{k-1} + 1}{2} \right\rfloor \right),$$

(58)

and the term $\left\lfloor x \right\rfloor$ denotes the largest integer smaller than or equal to $x$. The term

$$\text{erfc}\left((2i + 1)\sqrt{3\log_2 M E_b / 2(M - 1)}\right) = 2Q\left(\sqrt{2(2i + 1)} \sqrt{3\log_2 M E_b / 2(M - 1)}\right),$$

(59)

in Equation 57, when written in terms of the $Q(\cdot)$ function,
can be seen as twice the probability that the noise exceeds

\[(2i + 1) \sqrt{\frac{3\log_2 ME_b}{(M - 1)}}. \tag{61}\]

Thus, for the proposed impulsive noise models, that probability, given by

\[2\text{Prob} \left\{ \eta \geq (2i + 1) \sqrt{\frac{3\log_2 ME_b}{(M - 1)}} \right\}, \tag{62}\]

can be obtained by the integration of the pdf of the process that represents the noise, \(\eta(t)\), in the interval \((2i + 1) \sqrt{\frac{3\log_2 ME_b}{(M - 1)}} \rightarrow \infty\) \[31], \[32]. Then, one can write the probability \(P_b(k)\) for each one of the models presented in Sections III, IV, V and VI. Hence, substituting those probabilities in Expression 56, one obtain the expressions for the BEP of each one of the proposed models. It is worth to mention that the authors have successfully used this approach to determine the BEP of modulation schemes in Nakagami-\(m\), \(\eta - \mu\) and \(\kappa - \mu\) fading channels \[15], \[17], \[18]\.

A. BEP of the Gated Binary Gaussian Impulsive Noise

For the gated binary Gaussian impulsive noise, presented in Section III, the probability \(P_b(k)\) can be written as

\[P_b(k) = \frac{2}{\sqrt{M}} \sum_{i=0}^{(1-2^k)\sqrt{M}-1} u(i, k, M)\times \left( \alpha p_2 Q \left( \sqrt{a(i, M) \frac{\gamma g \gamma_i}{\gamma_g + \gamma_i}} \right) \right. \tag{63}\]

\[+ \left. (1 - \alpha p_2) Q \left( \sqrt{a(i, M) \gamma_g} \right) \right),\]

in which \(a(i, M) = 4(i+1)^2 \log_2(M)\), \(\gamma_g = E_b/N_0\) represents the signal-to-permanent noise ratio and the term \(\gamma_i = E_b/N_i\) will be referred to as signal-to-impulsive noise ratio, even knowing that the impulsive noise, in the general case, is modeled by the product \(C_1(t)C_2(t)\eta_i(t)\).

Figure 5 shows BEP curves for 64-QAM under gated binary Gaussian impulsive noise for different values of signal-to-impulsive noise ratio \(\gamma_i = E_b/N_i\). The probability distribution of the levels of amplitudes of \(C_2(t)\) and the value of the parameter \(\alpha\) are given by

\[\begin{cases} 
P\{C_2(t) = 1\} = p_2 = 0.7 \\
P\{C_2(t) = 0\} = 1 - p_2 = 0.3 \\
\alpha = 0.5 \end{cases}. \tag{64}\]

In Expression 64, \(\alpha = 0.5\) is the percentage of the duration \(T_2\), of the pulses \(P_{R_2}(t)\) of \(C_2(t)\), in which \(C_2(t)\) assumes unitary value. In this interval, the noise \(\eta_i(t)\) is added to the permanent noise with probability \(p_2\) or is absent with probability \(1 - p_2\). The case in which \(p_2 = 0.7\) characterizes a model in which the noisy pulses are present more frequently. It is observed in the curves of Figure 5 that when the signal-to-permanent noise reaches values greater than the signal-to-impulsive noise ratio \(\gamma_i\), for fixed values of \(\gamma_i\), the BEP tends to decrease. This occurs because when \(\gamma_g > \gamma_i\) the energy of the permanent noise \(\eta_g(t)\) is smaller than the energy of the impulsive noise \(C_2(t)\eta_i(t)\), which acts less frequently.

A behavior observed in the curves, in accordance with the BEP equations presented in this paper, is the tendency that the curves associated to the smaller value of \(\gamma_i\) remain constant with the increase of \(\gamma_g\). When \(\gamma_g\) is much greater than \(\gamma_i\), one of the functions \(Q(x)\) depends more on \(\gamma_i\) while the other function \(Q(x)\), which depends only on \(\gamma_g\), approaches zero. This explains the reason why the BEP tends to remain constant.

B. BEP of the Double Gated Binary Gaussian Impulsive Noise

For the double gated binary Gaussian impulsive noise, presented in Section IV, the probability \(P_b(k)\) can be written as

\[P_b(k) = \frac{2}{\sqrt{M}} \sum_{i=0}^{(1-2^k)\sqrt{M}-1} u(i, k, M)\times \left( \alpha \beta p_1 p_2 Q \left( \sqrt{a(i, M) \frac{\gamma_g \gamma_i}{\gamma_g + \gamma_i}} \right) \right. \tag{65}\]

\[+ \left. (1 - \alpha \beta p_1 p_2) Q \left( \sqrt{a(i, M) \gamma_g} \right) \right),\]

Figure 6 presents BEP curves for 64-QAM under double gated binary Gaussian impulsive noise, for different values of the signal-to-impulsive noise ratio, \(\gamma_i = E_b/N_i\). The probability distributions of the levels of amplitude of the signals \(C_1(t)\) and \(C_2(t)\) and the values of the parameters \(\alpha\) and \(\beta\) are given by

\[\begin{cases} 
P\{C_1(t) = 1\} = p_1 = 0.25 \\
P\{C_1(t) = 0\} = 1 - p_1 = 0.75 \\
\alpha = 0.5 \end{cases}. \tag{66}\]

and

\[\begin{cases} 
P\{C_2(t) = 1\} = p_2 = 0.75 \\
P\{C_2(t) = 0\} = 1 - p_2 = 0.25 \\
\beta = 0.5 \end{cases}. \tag{67}\]

In Expressions 67 and 66, \(\alpha = 0.5\) and \(\beta = 0.5\) represent, respectively, the percentage of the durations \(T_1\) and \(T_2\) of the pulses \(P_{R_1}(t)\) and \(P_{R_2}(t)\), of the modulating signals \(C_1(t)\) and
in Section V, the probability \( P \) kept constant and much smaller than \( \gamma \) energy of the impulsive noise is higher than the energy of the \( C \) distribution of the levels of amplitude of \( \gamma \) signal-to-impulsive noise ratio \( \gamma \). multilevel Gaussian impulsive noise for different values of \( \gamma \).

An aspect of Figure 6 to be pointed out, with respect for instance to the curve of \( \gamma_i = 10 \text{ dB} \), is that when \( \gamma_i \) is kept constant and much smaller than \( \gamma_g \) (case in which the energy of the impulsive noise is higher than the energy of the permanent noise), the BEP does not decrease with the increase of \( \gamma_g \).

**C. BEP of the Gated Multilevel Gaussian Impulsive Noise**

For the gated multilevel Gaussian impulsive noise, presented in Section V, the probability \( P_b(k) \) can be written as

\[
P_b(k) = \frac{2}{\sqrt{M}} \sum_{i=0}^{(1-2^{-t})\sqrt{M}-1} w(i, k, M) \times \left\{ \begin{array}{l}
\alpha \sum_{l=0}^{m_2-1} p_l Q\left( \sqrt{\alpha(i, M) - \gamma_i + c_l g} \right) \\
+ (1 - \alpha) Q\left( \sqrt{\alpha(i, M) g} \right)
\end{array} \right. 
\]

(68)

Figure 7 presents curves of BEP for 64-QAM under gated multilevel Gaussian impulsive noise for different values of signal-to-impulsive noise ratio \( \gamma_i = \frac{\gamma_0}{M} \). The probability distribution of the levels of amplitude of \( C_2(t) \) is given by

\[
\begin{align*}
\{ P(C_2(t) = 0) = p_{20} = 0.25 \\
\{ P(C_2(t) = 1) = p_{21} = 0.25 \\
\{ P(C_2(t) = 2) = p_{22} = 0.20 \\
\{ P(C_2(t) = 3) = p_{23} = 0.15 \\
\{ P(C_2(t) = 4) = p_{24} = 0.15
\end{align*}
\]

(69)

For such case, the levels of amplitude 0 and 1 are more probable to occur than levels 3 and 4, which have probability 0.15. That behavior of \( C_2(t) \) makes the modulating signal \( C_2(t) \eta(t) \) (which represents the component of the impulsive noise) have variations of higher amplitudes with a smaller probability and small variations of amplitudes with a higher probability. It is also observed that for \( \gamma_g = 25 \text{ dB} \) the BEP obtained is \( 10^{-3} \) with \( \gamma_i \) equals 25 dB. In such case, for \( \gamma_g = \gamma_i, N_t = N_0 \) and since \( \eta(t) \) affects the transmitted signal less frequently, it follows that the BEP can attain smaller values.

**D. BEP of the Double Gated Multilevel Gaussian Impulsive Noise**

For the double gated multilevel Gaussian impulsive noise, presented in Section VI, the probability \( P_b(k) \) can be written as

\[
P_b(k) = \frac{2}{\sqrt{M}} \sum_{i=0}^{(1-2^{-t})\sqrt{M}-1} w(i, k, M) \times \left\{ \begin{array}{l}
\alpha \sum_{m=0}^{m_1-1} \sum_{n=0}^{m_2-1} p_{1m} p_{2n} Q\left( \sqrt{\alpha(i, M) - \gamma_i + c_{1m} c_{2n} g} \right) \\
+ (1 - \alpha) Q\left( \sqrt{\alpha(i, M) g} \right)
\end{array} \right. 
\]

(70)

In Figure 8, the BEP curves are presented for the case in which the probability of \( C_1(t) \) assume null value is higher than the one of assuming unitary value. This is the case for which the modulated impulsive noise, with null amplitude, is more frequent. The probability distribution function of \( C_1(t) \) and \( C_2(t) \) for this case is

\[
\begin{align*}
\{ P(C_1(t) = 0) = p_{10} = 0.7 \\
\{ P(C_1(t) = 1) = p_{11} = 0.3 \\
\{ P(C_2(t) = 0) = p_{20} = 0.4 \\
\{ P(C_2(t) = 1) = p_{21} = 0.2 \\
\{ P(C_2(t) = 2) = p_{22} = 0.2 \\
\{ P(C_2(t) = 3) = p_{23} = 0.1 \\
\end{align*}
\]

(71)

It is observed in Figure 8, with respect to the curve corresponding to \( \gamma_i = 10 \text{ dB} \), the decrease of BEP for \( \gamma_g = 30 \text{ dB} \). That decrease is due to the probability distribution function of \( C_1(t) \) and \( C_2(t) \), or more precisely, due to the fact that the probability of \( C_1(t) = 0 \) is higher than the one of \( C_1(t) = 1 \).
Fig. 8. BEP of 64-QAM under double gated multilevel Gaussian impulsive noise.

Additionally, in the range where \( C_1(t) = 1 \), \( C_2(t) \) assumes null value with probability 0.4, which characterizes a more smooth influence of the modulated impulsive noise.

VIII. CONCLUSION

In this paper, an approach was presented for the study of gated Gaussian impulsive noise and its impact in the performance of a MAP receiver with M-QAM. In the approach, the presence or absence of the impulsive noise is characterized by an auxiliary random process which modulates the amplitude of the model AWGN \( \eta_i(t) \). That modulation of the process \( \eta_i(t) \) characterizes both its random variation of power and random variation of occurrence. Although the model of gated Gaussian impulsive noise already exists in the literature, its mathematical analysis by means of auxiliary processes \( C_1(t) \) and \( C_2(t) \) simplifies the calculus of the pdf, the autocorrelation and the power spectral density as well as the evaluation of its effects in the performance of receivers for a variety of digital modulation schemes, such as M-QAM. In this scenario, in the present paper, new exact expressions were presented for the BEP of M-QAM under impulsive noise.

As future work, one can cite the extension of the analysis of BEP presented here in order to account the fading [33] to fully characterize the effect of the gated noise in mobile communication systems.

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REFERENCES


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