BLOCK CODES FORMED FROM CATASTROPHIC CONVOLUTIONAL ENCODERS : APPLICATIONS TO BCM

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Resumo - Este artigo descreve a construção de códigos de bloco derivados da terminação da treliça de um codificador convolucional binário e *catastrófico*. Dois métodos de construção são considerados : o método Zero Tail (ZT) e um método Tail Biting (TB) *modificado*. Esquemas de modulação 4 - PSK codificada de bloco foram projetados baseados nos códigos construídos. Estes esquemas demonstraram um desempenho melhor do que esquemas derivados dos melhores codificadores não catastróficos.

Abstract - This article describes the construction of block codes derived from trellis termination of a binary *catastrophic* convolutional encoder. Two construction methods are considered: the Zero Tail (ZT) method and a *modified* Tail Biting (TB) method. 4-PSK block coded modulation (BCM) schemes are designed based on the constructed codes. These schemes outperform similar schemes derived from the best non-catastrophic encoder.

Keywords: Catastrophic convolutional encoders, Trellis termination, Zero Tail and Tail Biting methods, Block Coded Modulation.

1. INTRODUCTION

Methods for conversion of convolutional codes into block codes have been considered in the literature [1, 2, 3]. In all of these methods, *non*-catastrophic convolutional encoders were considered. Not all of these methods can be applied in a simple way to the case when the convolutional encoder is catastrophic. It is known that block codes derived from catastrophic convolutional encoders may have row distance[1] , d^r, greater than that derived from the best *non*-catastrophic encoders (with the same rate and constraint length,$m+1$). A couple of these catastrophic encoders of rate $1/2$ are listed in Tab. 1 [4]. Note that this table lists the catastrophic encoders for values of m for which there is no non-catastrophic encoder whose corresponding distance d^r reaches the Heller bound (For $m = 9$, an exhaustive search indicated that there are non-catastrophic encoders whose distance d^r reach the Heller bound, eg the encoder with generators in octal form 4534 and 6364 has $d_{free} = d^r = 13$) .

m	Generators (in octal)	d^r
4	72 and 56	8
5	no one	9
7	no one	11
11	6731 and 5237	16

Table 1: Some catastrophic encoders of rate $1/2$ whose row distance reaches the Heller bound.

The fact that block codes derived from catastrophic encoders may have optimal row distances would justify a more detailed investigation on these encoders. There are also other reasons that justifies such investigation: 1) The set of catastrophic encoders should be only a small subset of the set of all convolutional encoders [5], p. 308. 2) It was recently shown [6], that the iterative construction of Reed-Muller (RM) codes [7] can be some times defined through trellis termination of a *catastrophic* convolutional encoder using the ZT method.

It is worth to notice that any convolutional encoder with termination determines a simple encoder for the derived block code and also suggests a maximum likelihood decoding algorithm, which can be implemented by applying the Viterbi algorithm to the terminated trellis [6].

2. CATASTROPHIC ENCODERS

A convolutional encoder is said to be catastrophic when the input sequence of infinite weight, generates an output sequence of finite weight (Hamming or Euclidian) [8]. This means that a finite number of errors in a discrete output channel may cause an infinite number of errors in the decoded information bits.

Consider the convolutional encoder (n,k,m), where the parameters represent, respectively, the number of outputs, of inputs and of memory units of the encoder. An algebraic way to check if a (n,1,m) encoder is catastrophic, is by otaining the Greatest Common Divisor (GCD) of its generator polynomials [5]. If the GCD is not a power of D, D^l, where l is a natural number, then the encoder is catastrophic.

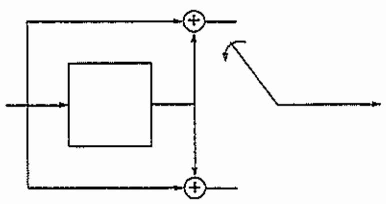

Figure 1: Convolutional encoder (2,1,1).

Example 1: Consider the encoder (n,k,m)=(2,1,1). Its gener-

[1]Weight of the minimum weight path that diverges from the zero state and later reconverges to this state [4]

ator polynomials are $g^{(1)} = 1 + D$ and $g^{(2)} = 1 + D$ (see Fig. 1). The GCD of $g^{(1)}$ and $g^{(2)}$ is 1+D, which is not a power of D. Therefore, the encoder is catastrophic.

Another way to check whether an encoder is catastrophic or not, is to analyze its state diagram. If is there any zero weight loop, with the exception of the zero state self-loop, then the encoder is catastrophic.

Example 1 (continued): Consider the state diagram of the encoder of Example 1 represented in Fig. 2, where the convention represents input bit/output bits. It is observed that there is a self-loop over state 1 and therefore the encoder is catastrophic.

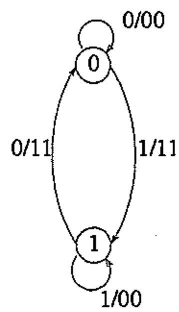

Figure 2: State diagram.

3. TERMINATION OF TRELLIS DIAGRAMS

In the following, two methods for deriving block codes from a convolutional encoder are described [3]:

Zero Tail (ZT): The codewords of the binary linear code formed by this method are all of the output sequences of the encoder when the encoder is initialized to the all-zero state and $(K - m)$ arbitrary data bits are input into the encoder followed by $m\ k$ zeros, where K is greater than m. The rate of the resultant block code is $\frac{k\,(K-m)}{n.K}$, where m is now defined as the maximum length of all k shift registers.

The codewords of the derived block code can be represented by paths in the trellis diagram (the state diagram representation in time) of the encoder. The trellis termination of encoder of Example 1 for the ZT method is shown in Fig. 3.

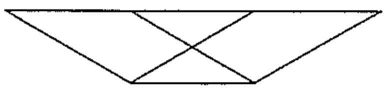

Figure 3: Zero Tail or ZT termination.

Tail Biting (TB): The codewords of the binary linear code formed by this method are all output sequences of the encoder when the encoder is initialized to the corresponding last $m\ k$ bits of an arbitrary K bits sequence and then those K bits are input into the encoder. The rate of the resultant block code is $\frac{k}{n}$.

The trellis termination of encoder of Example 1 for the TB method is shown in Fig. 4.

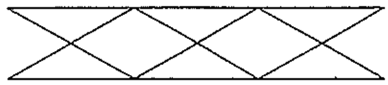

Figure 4: Tail Biting or TB termination.

Consider the encoder trellis of Example 1 with the cut as shown in Fig. 5. The resultant block code is a $(N,K,d_H) =$ (16,7,4), where N, K and d_H are the block size, the number of information bits and the minimum Hamming distance, respectively. On a similar way, varying only the instant of the cut, Tab. 2 is obtained. Note that the code rate gets closer to $\frac{1}{2}$ (which is the rate of the convolutional code) as the code size increases.

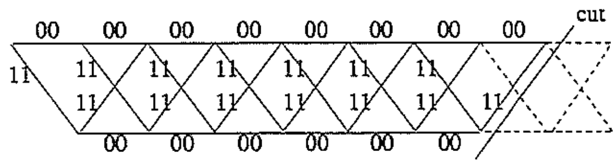

Figure 5: Trellis terminated.

It is important to notice that the trellis termination breaks the catastrophic behavior, as it limits the number of error events [9].

A very useful tool in the analysis of trellis termination of convolutional encoders is the A matrix described as follows. Its a_{ij} elements are given by D^h if is there an input that takes the encoder from state i to state j and that produces an output of weight h, otherwise they are considered to be zero [3].

Example 1 (continued): Fig. 6 shows the A matrix of the encoder. It is easy to check its elements through the state diagram of Fig. 2.

The a_{ij}^K element of the A^K matrix represents the weight of codewords that start in state i and end in state j when K bits are input into the encoder. For the case of a two memory units convolutional encoder, the weight distributions of binary linear codes formed from the ZT and TB methods, are given by the sum of checked terms of A^K in Fig. 7 and Fig. 8, respectively [3].

The reader may have already noticed in Fig. 4 that there is a problem with the use of TB method for a catastrophic encoder: distinct information bits sequences may result in the same codeword. The information sequence 1,1,1 generates the all zero codeword if the encoder was in state "1". And the information sequence 0,0,0 also generates the all zero codeword if the encoder was in state "0". If we would exclude all codewords generated from state "1" we would not have this problem, but the resultant code would be the same one

Code (N,K,d_H)	Rate, R=K/N
(16,7,4)	7/16=0,438
(18,8,4)	8/18=0,444
(20,9,4)	9/20=0,450
(22,10,4)	10/22=0,455

Table 2: Comparative table.

$$A = \begin{bmatrix} 1 & D^2 \\ D^2 & 1 \end{bmatrix}$$

Figure 6: A matrix of the encoder of Example 1.

resulted from the use of ZT method. This exclusion corresponds in matrix A^K to the exclusion of element a_{22}^K of the main diagonal from the sum that results in the weight distribution. Fortunately, for encoders with $m\ k > 1$, it is possible to exclude some terms of the main diagonal of A^K and the code resulted would not be the same as that obtained by using ZT method. This exclusion of terms of the main diagonal is called *modified TB*. The Theorem in the Appendix shows that with an appropriate exclusion, the derived block code is linear.

$$\begin{bmatrix} x & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}$$

Figure 7: Terms that contribute to the weight distribution for the ZT method, with m=2.

4. SEARCH FOR GOOD CONVOLUTIONAL ENCODERS

Based on the idea of getting catastrophic encoders, an exhaustive search has taken place for $m = 4$ and rate $1/2$. The algorithm proposed by Larsen [4], which calculates the d^r of an (n,1,m) encoder, was used. In fact this algorithm is a corrected version of the algorithm proposed by Bahl et al [10].

One of the two encoders that came out was the one with generator polynomials $g^{(1)} = 1 + D + D^2 + D^4$, $g^{(2)} = 1 + D^2 + D^3 + D^4$ and $d^r = 8$, which is the same code listed in [11]. The best non-catastrophic encoder, with the same rate and constraint length, has $d^r = 7$. Excluding the codewords corresponding to the elements $a_{ii}^K, 9 \leq i \leq 2^4$, the remaining codewords form a linear block code (the states of matrix A are in the following order: $\{0000, 0001, \cdots, 1110, 1111\}$).

Fig. 9 shows the encoder and Fig. 10 the trellis representation of the encoder. The termination of this trellis in instant 8 is shown in Fig. 11. Note that this terminated code is a (16,4,8) block code. With the use of a coset it is possible to obtain the (16,5,8) code, which is a Reed-Muller code.

$$\begin{bmatrix} x & - & - & - \\ - & x & - & - \\ - & - & x & - \\ - & - & - & x \end{bmatrix}$$

Figure 8: Terms that contribute to the weight distribution for the TB method, with m=2.

This and other Reed-Muller codes are the best block codes for fixed K,d_H and $N \leq 32$ [7].

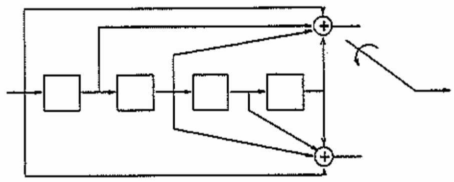

Figure 9: Encoder of the obtained code.

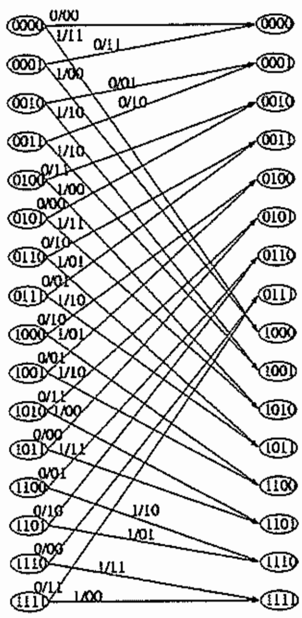

Figure 10: Trellis of the obtained code.

5. APPLICATIONS TO BCM

Fig. 12 shows the BCM scheme used. The mapping of the c^0, c^1 bits is shown in Tab. 3. This mapping is chosen in such a way that the quadratic Euclidean distance, $d_E{}^2(S, \hat{S})$, between two 4-PSK signals S and $\hat{S}$, is proportional to the Hamming distance, $d_H(c, \hat{c})$, between the corresponding pairs of coded bits $\underline{c}$=(c^0,c^1), $\hat{c}$=($\hat{c}^0$,$\hat{c}^1$). It is easy to show that $d_E{}^2(S, \hat{S})$=$2E_s d_H$ ($\underline{c}$,$\hat{c}$), where E_s is the average energy of the 4-PSK constellation's signals. This implies that obtaining a code with the maximum distance d_H is the same as obtaining a corresponding code in the Euclidean space with maximum d_{min} (Euclidean minimum distance).

c^1c^0	S
0 0	S_0
0 1	S_1
1 0	S_3
1 1	S_2

Table 3: Mapping of bits in signal space.

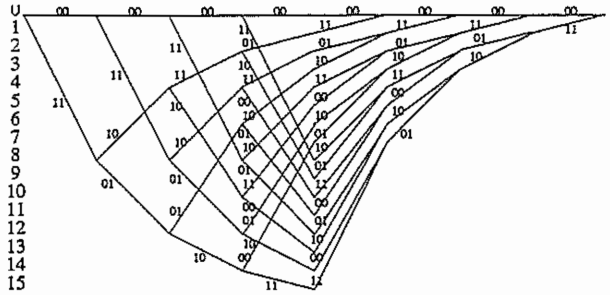

Figure 11: ZT terminated trellis with 8 intervals.

With this bit mapping and the ZT terminated trellis as shown in Fig. 11, a BCM scheme with d^2_{min}=16E_s is obtained. This scheme will be called Scheme I. The Euclidean weight distribution of the BCM scheme can be obtained by modifying the matrix method suggested in [3].

It would be interesting now to compare the ZT termination obtained here for a catastrophic encoder with the termination of the best non-catastrophic encoder with $n = 2$ and $m = 4$, for which $d^r = 7$ [5]. For $K = 8$, the BCM scheme derived from trellis termination of this encoder (here called Scheme II) has d^2_{min}=14E_s. Therefore, an asymptotic gain of $10log_{10}\frac{16}{14} = 0.58$ dB of Scheme I over Scheme II is expected.

Fig. 13 shows the performance of the two schemes as a function of the signal-to-noise ratio $\frac{E_b}{N_0}$ (E_b is the energy per bit and $N_0/2$ is the bilateral power density of the AWGN noise). In Fig. 13, the block error probability $P(e)$ is given by the approximation

$$P(e) \simeq N(d_{min}).\frac{1}{2}.erfc(\frac{d_{min}}{2\sqrt{N_0}}),$$

where $erfc(.)$ is the complementary error function, and $N(d_{min})$ is the number of sequences with distance d_{min}. From Fig. 13, a gain of almost 0.5 dB for $P(e) = 10^{-8}$ of Scheme I over Scheme II can be expected.

Using the same procedures described before, four schemes for $K = 12$ were analyzed:

- Scheme III - uses ZT method with the non-catastrophic encoder of $d^r = 7$.
- Scheme IV - uses ZT method with the catastrophic encoder of $d^r = 8$.
- Scheme V - uses TB method with the non-catastrophic encoder of $d^r = 7$.

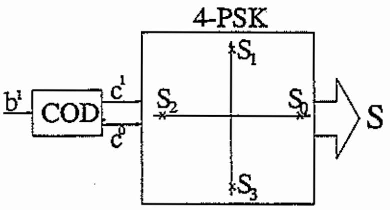

Figure 12: BCM scheme used.

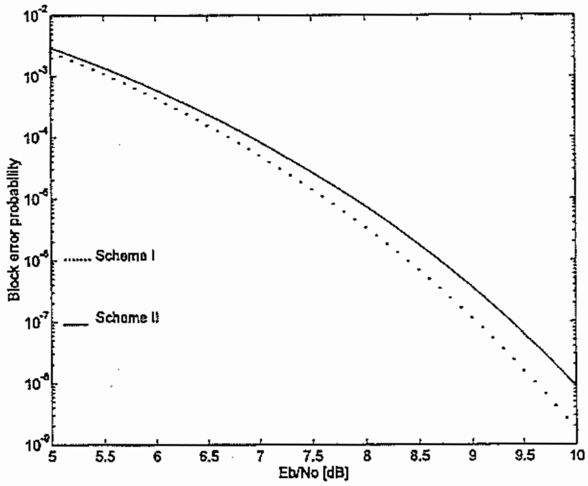

Figure 13: Error performance of Schemes I and II on an AWGN channel.

- Scheme VI - uses the modified TB method with the catastrophic encoder $d^r = 8$.

Fig. 14 shows the performance of Schemes III, IV, V and VI as a function of $\frac{E_b}{N_0}$. It can be seen in Fig. 14 that Scheme IV behaves almost like Scheme V. It can also be seen in Fig. 14 that the modified TB method (Scheme VI) shows a better performance than the traditional TB method (Scheme V) for $\frac{E_b}{N_0} > 5.0$ dB. It is worth to notice that Scheme V has a rate of 1 bit per 4 - PSK symbol while Scheme VI has a rate of 11/12 bit per 4 - PSK symbol. The Theorem in the Appendix shows that applying the modified TB method to a class of binary catastrophic encoders of rate $\frac{k}{n}$ results in a block code of rate $\frac{kK-1}{nK}$.

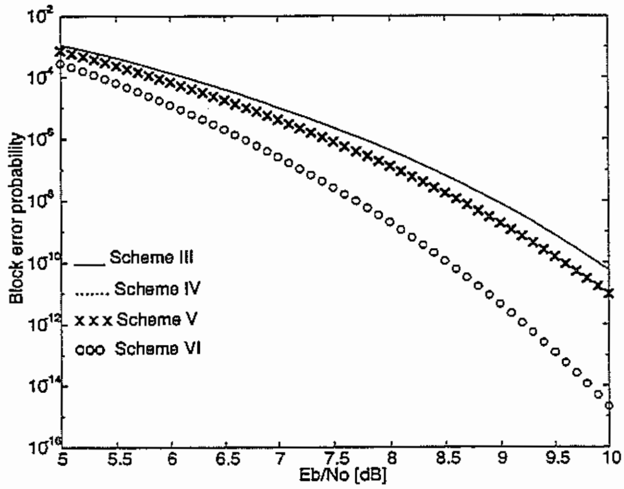

Figure 14: Error performance of Schemes III, IV, V and VI on an AWGN channel.

For the sake of completeness Tab. 4 shows the weight distribution for Schemes I and II and Tab. 5 shows the weight distribution for Schemes III, IV, V and VI. Performance results in a Rayleigh fading channel for the schemes given here are described in [12].

Scheme	Weight Enumerator
I	$1+13\,D^{16}+2\,D^{24}$
II	$1+5\,D^{14}+3\,D^{16}+2\,D^{18}+4\,D^{20}+$ $+D^{22}$

Table 4: Euclidean distance weight enumerators of schemes with K=8.

Scheme	Weight Enumerator
III	$1+13\,D^{14}+12\,D^{16}+12\,D^{18}+\ldots+$ $+6\,D^{34}+2\,D^{36}+D^{38}$
IV	$1+50\,D^{16}+152\,D^{24}+53\,D^{32}$
V	$1+12\,D^{10}+30\,D^{12}+84\,D^{14}+\ldots+$ $+84\,D^{34}+38\,D^{36}+12\,D^{38}$
VI	$1+375\,D^{16}+1296\,D^{24}+375\,D^{32}+D^{48}$

Table 5: Euclidean distance weigh enumerators of schemes with $K=12$.

6. FINAL REMARKS AND CONCLUSIONS

In [13], a trellis coded modulation (TCM) scheme with rate equal to 1 bit per 4-PSK symbol is given, whose constituent non-catastrophic binary convolutional code has d_{free}= 8 and rate 2/4. Other TCM schemes with the same rate and d_{free} given in the recent literature are also referenced in [13] (two of them utilize time-invariant convolutional encoders with rates 2/4 and 4/8 and one of them a time-varying encoder with rate 1/2). 4-PSK BCM schemes derived from these encoders should also have rate equal to 1 bit per symbol. As described in Section 4 for the $(16,4,8)$ code, we believe that it is possible to add an appropiate coset to the code of Scheme VI in order to obtain a scheme with rate equal to 1 bit per 4-PSK symbol. The decoding process for this new scheme could be implemented with two identical trellises by applying the same algorithm to both trellises in parallel. This property can be advantageous if a fast decoding process is needed.

The article considered two construction methods for obtaining linear block codes derived from a catastrophic convolutional encoder: the ZT method and the TB method. In contrast to the ZT method, the TB method cannot be applied in a simple way to the case when the convolutional encoder is catastrophic. A modified TB construction method was then suggested. The performance of several 4-PSK BCM schemes constructed from block codes formed from convolutional encoders were analysed. The schemes formed from catastrophic encoders have a better performance when compared with schemes formed from non-catastrophic encoders. For this reason, catastrophic encoders deserve consideration in the design of these schemes. A better study on the modified TB method is under development.

7. APPENDIX

Theorem: Consider the class of binary catastrophic convolutional encoders whose augmented state diagram has only one zero Hamming weight loop: the self-loop around the all-one state. The code words corresponding to the sum of half of the main diagonal elements a_{ii}^K of A^K, form a linear block code with rate $\frac{k}{n}\frac{K-1}{K}$.

Proof:

In the following, we consider a model for the encoder with k feedback-free shift registers of equal length m. We assume that some of the tap gains can be equal to zero. Let $\mathbf{u}$ denote the input sequence of length $K\ k=(L+m)\ k$. $\mathbf{u}$ is of the form:

$$\mathbf{u}=(u_0,u_1,\ldots,u_{Lk-1},s_0,s_1,\ldots,s_{mk-1}),$$

where $s_0,s_1,\ldots,s_{mk-1}$ represents the initial state of the encoder. We form the extended input sequence $\mathbf{u}^0$,

$$\mathbf{u}^0 = (s_0,s_1,\ldots,s_{mk-1},u_0,u_1,\ldots,u_{Lk-1},s_0,s_1,\ldots,s_{mk-1}),$$

which takes the encoder from all-zero state $s_0=s_1=\ldots=s_{mk-1}=0$, at time $-m$ to state $s_0,s_1,\ldots,s_{mk-1}$ at time $L+m$, passing through the same state $s_0,s_1,\ldots,s_{mk-1}$ at time zero.

Let G be the generator matrix of the convolutional code. The output sequences $\mathbf{v}$ generated by the TB method are obtained from the sequences $\mathbf{v}^0$,

$$\mathbf{v}^0=\mathbf{u}^0\,G,$$

by deleting the first $(mk)n$ components. Therefore, the sequences generated by the TB method are the union of 2^{mk} cosets. The code generated by the ZT method is the zero coset (obtained by setting the encoder state to the all-zero state).

Let $\mathbf{u}'$ denote the complementary sequence of $\mathbf{u}$, i.e., $u_i'=u_i+1$, where $+$ is a modulo 2 sum. Since $\mathbf{u}$ and $\mathbf{u}'$ determine two complementary sequences of encoder states and the encoder has a self-loop around the all-one state (as stated in theorem), then,

$$\mathbf{v}(=\mathbf{u}G)+\mathbf{v}'(=\mathbf{u}'G)=\mathbf{0},$$

which implies that $\mathbf{v}=\mathbf{v}'$. This means that the cosets generated by representatives of two initial complementary states are the same. However, there is only one zero weight loop. Therefore, if one of the two identical cosets corresponding to each pair of complementary states, is excluded, then, the remaining set form a block code with only one information bit less than a block code formed from the conventional TB method. By choosing the code words corresponding to the sum of half of the main diagonal elements a_{ii}^K of A^K, whose initial states are labeled by $s_0=0,s_1,\ldots,s_{mk-1}$, then, the derived block code is linear.

Q.E.D.

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