SPACE-TIME CONVOLUTIONAL CODES OVER GF(p) FOR THE QUASI-STATIC, FLAT RAYLEIGH FADING CHANNEL

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Abstract - In this paper, we consider a space-time coded system consisting of a rate $R = 1/n$ linear convolutional encoder over GF(p), p a prime, followed by n mappers from GF(p) into a p-ary signal constellation, and by a transmitter with n transmit antennas. At each time, the n p-ary coded symbols are transmitted simultaneously from the n antennas. The convolutional codes are designed to provide the best error performance in the quasi-static, flat Rayleigh fading channel, according to the rank and the determinant criteria derived by Tarokh et al. A spectral efficiency of $\log_2(p)$ b/s/Hz is achieved. Simple conditions on the generator matrices of a rate $R = 1/2$ convolutional code are given so that maximum diversity advantage is guaranteed. The linear structure of these convolutional codes reduces significantly the computer search effort with respect to the determinant criterion. In this regard, some properties of reciprocal STC's are derived. New space-time codes with two transmit antennas are presented for the 5-PSK and 7-PSK modulations, with spectral efficiencies 2.32 and 2.81 b/s/Hz, respectively. Simulation results in the form of frame error rate versus signal-to-noise ratio are reported for these codes, showing agreement with the theoretical results.

Keywords: Diversity, fading channels, multiple transmit antennas, space-time convolutional codes, wireless communications.

Resumo - Neste trabalho consideramos um sistema codificado espácio-temporal composto por um codificador convolucional linear de taxa $R = 1/n$ sobre GF(p), onde p é primo, seguido de n mapeadores de símbolos em GF(p) para uma constelação de sinais p-ários e de um transmissor contendo n antenas transmissoras. Em cada instante de tempo, os n símbolos p-ários codificados são transmitidos simultaneamente através das n antenas. Os códigos convolucionais são projetados para proporcionar o melhor desempenho de erro em um canal com desvanecimento Rayleigh plano quase-estático, de acordo com os critérios do posto e do determinante derivados por Tarokh et al. Uma eficiência espectral de $\log_2(p)$ b/s/Hz é alcançada. Restrições simples impostas às matrizes geradoras de um codificador convolucional de taxa $R = 1/2$ são apresentadas para garantir ganho de diversidade máximo. A estrutura linear destes códigos re-

duz significantemente o esforço da busca computacional com relação ao critério do determinante. Nesta direção, algumas propriedades de STC recíprocos são demonstradas. Novos códigos espácio-temporais com duas antenas transmissoras são apresentados para as modulações 5-PSK e 7-PSK com eficiências espectrais de 2.32 e 2.81 b/s/Hz, respectivamente. Resultados de simulações na forma de taxa de erro de *frame* versus relação sinal-ruído são apresentados e estão de acordo com os resultados teóricos.

Palavras-chave: Diversidade, canais com desvanecimento, múltiplas antenas transmissoras, códigos convolucionais espácio-temporais, comunicações sem fio.

1. INTRODUCTION

In a recent wave of information theoretic results, Telatar [1], Foschini and Gans [2], and Marzetta and Hochwald [3] have demonstrated that the capacity of wireless communications systems employing multiple antennas is noticeably greater than the capacity of single-antenna systems for fading channels. Motivated by [1] and [2], Tarokh et al. [4] introduced the so-called "space-time coding," whereby redundancy is placed in an appropriate way across space (transmit antennas) as well as time to achieve coding and transmit diversity advantages simultaneously. Space-time coding has rapidly become the most accepted technique for providing reliable communications over fading channels employing multiple transmit antennas.

A remarkable result derived in [4] is that for the quasi-static, flat Rayleigh fading channel the best space-time codes (STC) may be obtained by optimizing two design criteria, namely, the *rank* criterion and the *determinant* criterion. Some STC for two transmit antennas for the 4-PSK and 8-PSK modulations were provided in [4]. The STC presented in [4], and those presented subsequently in [5, 6], were either hand designed or found by exhaustive computer search. The main difficulty for designing STC is that the two aforementioned criteria apply to the complex field of baseband modulated signals rather than a discrete domain (*e.g.*, the binary Galois field) in which the underlying codes could be designed. Recently, important papers providing general design procedures of STC were written by Hammons and El Gamal [7, 8], by Blum [9, 10], and by Grimm et. al. [6]. In [7, 8], general binary rank criteria (in substitution to the complex-based rank criteria) were developed that are simpler and yet sufficient to guarantee that the associated STC achieve maximum diversity for any number of transmit antennas. Results are valid for the BPSK and QPSK modulations. Blum's papers deal with binary convolutional codes that serve as STC. Necessary and sufficient conditions are given in [9, 10] for the

STC to achieve maximum diversity gain. Also, Blum developed methods for calculating and bounding the coding gain (related to the determinant criterion). In [6] the design rules proposed in [4] for the case of two transmit antennas are generalized to the multiple antennas case through the concept of "zeros symmetry". In all the previous papers, the underlying codes were either binary or over an extended binary field.

In this paper, an alternative design procedure for achieving diversity and coding advantages is presented. We consider a space-time coded system consisting of a rate $R = 1/n$ linear convolutional encoder over GF(p), where p can be any prime number, followed by a serial-to-parallel converter, n mappers from GF(p) into a p-ary signal constellation (not necessarily PSK), and a transmitter with n transmit antennas. This approach differs from those in the papers referenced above in that here the coded symbols are in one-to-one correspondence to the signals from the constellations. This makes the code design and the code search much easier to cope with. A spectral efficiency of $\log_2(p)$ b/s/Hz is achieved. Such non-integral rates may be useful to serve a wider range of multimedia applications. Although the case of multiple transmit antennas can be considered (by means of Grimm's "zeros symmetry"), we specialize in this paper to the case of two transmit antennas. Simple conditions on the generator matrices of a rate $R = 1/2$ convolutional code are given so that maximum diversity advantage is guaranteed. Under these conditions, and due to the linear structure of the convolutional codes, the computer search effort with respect to the determinant criterion is dramatically reduced. In this regard, some properties of reciprocal STC's are derived. New space-time codes with two transmit antennas are presented for the 5-PSK and 7-PSK modulations, with spectral efficiencies 2.32 and 2.81 b/s/Hz, respectively.

This paper is organized as follows. The space-time system model and the two design criteria for the quasi-static, flat Rayleigh fading channel as presented in [4] are summarized in Section 2. In Section 3, space-time convolutional codes (STCC) over GF(p) are introduced, and sufficient conditions on the generator matrices of a rate $R = 1/2$ convolutional code are given so that maximum diversity advantage is guaranteed. Some examples of new STCC over GF(5) and GF(7) with encoder memories 1 and 2, and some properties of reciprocal STC's are presented in Section 4. Simulation results in the form of frame error rate versus signal-to-noise ratio are reported, in Section 5, for these codes for $m = 1, 2, 3$ receive antennas. The corresponding coding and diversity advantages are verified, in agreement with the theoretical results presented in Section 3. Finally, in Section 6 we draw some final comments.

2. SYSTEM MODEL

We consider a communication system employing n transmit antennas and m receive antennas, as shown in Fig. 1. At the transmitter, data is first encoded by a rate $R = 1/n$ convolutional encoder over GF(p), p a prime, whose output is divided into n parallel streams. These streams are mapped into a p-ary signal constellation and transmitted simultaneously from the n antennas. A space-time system has been
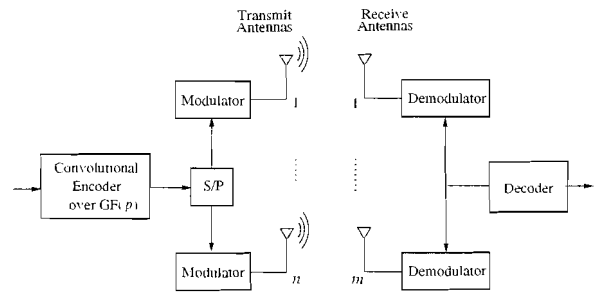

Figure 1. Space-time system model.

modeled by Tarokh *et. al.* [4] in such a way that the signal received by the jth antenna at time t, d_t^j, is given by

$$d_t^j = \sum_{i=1}^{n} \alpha_{i,j} c_t^i \sqrt{E_s} + \eta_t^j \qquad (1)$$

where c_t^i is the signal transmitted from the ith antenna at time t, E_S is the average energy of the transmitted signal, η_t^j is a zero-mean complex white Gaussian noise with variance $N_0/2$ per dimension, and $\alpha_{i,j}$ denotes the fade present in the path from the ith transmit antenna to the jth receive antenna. Under the Rayleigh fading assumption, the fades $\alpha_{i,j}$, for $1 \le i \le n$ and $1 \le j \le m$, are modeled as independent samples of a zero-mean complex Gaussian random process with variance 0.5 per dimension. In practice, to achieve independent fades the antennas must be separated by a few wavelengths. For the quasi-static, flat-fading channel, it is assumed that the path fades remain constant during a frame and change independently from one frame to another. Also, assume the receiver perfectly knows the channel state information (i.e., the α's) and that the Viterbi algorithm with the Euclidean metric is used in the decoder.

2.1 DESIGN CRITERIA

In this section, we briefly review the derivation of the design criteria for slow Rayleigh fading channels [4]. For the system in Fig. 1, and described by (1), the conditional pairwise error probability, defined as the probability that a maximum-likelihood decoder decides erroneously in favor of codeword

$$\mathbf{e} = e_1^1 e_1^2 \cdots e_1^n e_2^1 e_2^2 \cdots e_2^n \cdots e_l^1 e_l^2 \cdots e_l^n$$

when the correct codeword is:

$$\mathbf{c} = c_1^1 c_1^2 \cdots c_1^n c_2^1 c_2^2 \cdots c_2^n \cdots c_l^1 c_l^2 \cdots c_l^n,$$

given a fixed set of fades, can be approximated by:

$$\begin{aligned} P(\mathbf{c} \to \mathbf{e} | \alpha_{i,j}, i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m) \\ \le exp(-d^2(\mathbf{c}, \mathbf{e}) E_s / 4 N_0) \end{aligned} \qquad (2)$$

where $d^2(\mathbf{c}, \mathbf{e})$ is the square Euclidean distance between the sequences $\mathbf{c}$ and $\mathbf{e}$, and is given by:

$$d^2(\mathbf{c}, \mathbf{e}) = \sum_{j=1}^{m} \sum_{t=1}^{l} \left| \sum_{i=1}^{n} \alpha_{i,j}(c_t^i - e_t^i) \right|^2. \qquad (3)$$

After some algebraic manipulations, (3) can be written as:

$$d^2(\mathbf{c}, \mathbf{e}) = \sum_{j=1}^{m} \Omega_j A(\mathbf{c}, \mathbf{e}) \Omega_j^H. \tag{4}$$

where $\Omega_j = (\alpha_{1,j}, \alpha_{2,j}, \ldots, \alpha_{n,j})$ and $\{A(\mathbf{c}, \mathbf{e})\}_{pq} = \sum_{t=1}^{l} (c_t^p - e_t^p)(c_t^q - e_t^q)^*$, for $1 \leq p, q \leq n$. Here the Hermitian transposition is denoted by H and the complex conjugate by $*$. Note that the matrix $A(\mathbf{c}, \mathbf{e})$ is Hermitian and can be factored as $B(\mathbf{c}, \mathbf{e})B(\mathbf{c}, \mathbf{e})^H$, where

$$B(\mathbf{c}, \mathbf{e}) \triangleq \begin{pmatrix} e_1^1 - c_1^1 & e_2^1 - c_2^1 & \cdots & e_l^1 - c_l^1 \\ e_1^2 - c_1^2 & e_2^2 - c_2^2 & \cdots & e_l^2 - c_l^2 \\ \vdots & \vdots & \ddots & \vdots \\ e_1^n - c_1^n & e_2^n - c_2^n & \cdots & e_l^n - c_l^n \end{pmatrix}$$

is a square root of matrix $A(\mathbf{c}, \mathbf{e})$. According to [11], if a matrix A with a square root B is Hermitian then the eigenvalues of $A(\mathbf{c}, \mathbf{e})$ are non-negative real numbers.

Now, we write $d^2(\mathbf{c}, \mathbf{e})$ in terms of the eigenvalues of the matrix $A(\mathbf{c}, \mathbf{e})$. From linear algebra [11], for each Hermitian matrix A there exists a unity matrix V and a real diagonal matrix $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $VAV^H = D$, where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues of $A(\mathbf{c}, \mathbf{e})$. Defining $(\beta_{1,j}, \beta_{2,j}, \ldots, \beta_{n,j}) = \Omega_j V^*$, we arrive after some algebraic manipulation at the following expression for the upper bound in (2):

$$\prod_{j=1}^{m} exp(-(E_s/4N_0) \sum_{i=1}^{n} \lambda_i |\beta_{i,j}|^2). \tag{5}$$

Since the $\alpha_{i,j}$'s are modeled as independent samples of a zero-mean complex Gaussian random process with variance 0.5 per dimension, and V is unitary, then the $\beta_{i,j}$'s are also independent zero-mean complex Gaussian random variables with variance 0.5 per dimension. Therefore, we can calculate the average value of (5) and obtain, after some simplifications based on the large signal-to-noise ratios assumption, an upper bound on the average pairwise error probability, given by:

$$P(\mathbf{c} \rightarrow \mathbf{e}) \leq \left(\prod_{i=1}^{r} \lambda_i \right)^{-m} \left(\frac{E_s}{4N_0} \right)^{-rm}, \tag{6}$$

where r is the rank of the matrix $B(\mathbf{c}, \mathbf{e})$ and $\lambda_i, i = 1, \ldots, r$, are the nonzero eigenvalues of $A(\mathbf{c}, \mathbf{e})$.

Based on (6), Tarokh *et. al.* [4] arrived at two design criteria for the quasi-static, flat Rayleigh fading channel, namely,

- *The Rank Criterion:* In this criterion the parameter to be maximized is the minimum rank r of matrix $B(\mathbf{c}, \mathbf{e})$, over all distinct pairs of codewords $\mathbf{c}$ and $\mathbf{e}$. It is said that this is an r-space-time code. The diversity advantage is $rm \leq nm$, with equality at full rank condition, that is, $r = n$.

- *The Determinant Criterion:* For a given diversity advantage rm, the goal of this criterion is to maximize the minimum geometric mean of the nonzero eigenvalues of the matrix $A(\mathbf{c}, \mathbf{e})$, $(\lambda_1 \lambda_2 \ldots \lambda_r)^{\frac{1}{r}}$, over all distinct pairs of codewords $\mathbf{c}$ and $\mathbf{e}$. This represents the coding gain.
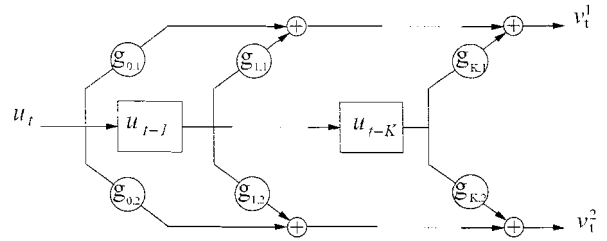

Figure 2. Rate 1/2 convolutional encoder over GF(p).

In the next section we derive conditions on the generator matrix of a rate $R = 1/2$ convolutional code so that maximum diversity advantage, as presented in Section 2, is guaranteed for the corresponding 2-space-time codes.

3. SPACE-TIME CONVOLUTIONAL CODES OVER GF(p)

Let $U(D) = u_0 + u_1 D + u_2 D^2 + \cdots$ be the polynomial over GF(p), p a prime, representing a data sequence. This sequence is encoded by a rate $R = 1/n$ convolutional encoder over GF(p), which is the direct realization of the polynomial generator vector:

$$\mathbf{G}(D) = [G_1(D), G_2(D), \ldots, G_n(D)],$$

producing the encoded vector

$$\mathbf{V}(D) = U(D)\mathbf{G}(D) = [V^1(D), V^2(D), \ldots, V^n(D)],$$

where $V^i(D) = v_0^i + v_1^i D + v_2^i D^2 + \cdots + v_t^i D^t + \cdots$, for $i = 1, 2, \ldots, n$, are the n encoded sequences. The code generators are $G_i(D) = g_{0,i} + g_{1,i}D + g_{2,i}D^2 \cdots + g_{K,i}D^K$, for $i = 1, 2, \ldots, n$, where K is the memory of the encoder. An example of a rate 1/2 convolutional encoder over GF(p) is shown in Fig. 2.

At each time t, a data symbol from GF(p) produces a block of n symbols from GF(p) denoted by $(v_t^1, v_t^2, \ldots, v_t^n)$. Regarding the space-time system model in Fig. 1, these symbols are mapped into a p-ary signal constellation, as indicated in Fig. 3 for the 5-PSK and 7-PSK constellations. The n complex symbols, $(c_t^1, c_t^2, \ldots, c_t^n)$, are then transmitted by the n antennas. A sequence of blocks $(c_t^1, c_t^2, \ldots, c_t^n)$, for $t = 1, 2, \ldots, l$ forms a codeword $\mathbf{c}$ of the space-time code.

From now on, we shall focus our attention on the case of $n = 2$ transmit antennas. Thus, according to Tarokh's design criteria, the code search should aim at finding the "full rank" code $\mathcal{C}$ that maximizes the minimum determinant

$$det \left(\sum_{i=1}^{l} (e_i^1 - c_i^1, e_i^2 - c_i^2)^* (e_i^1 - c_i^1, e_i^2 - c_i^2) \right), \tag{7}$$

over all pairs of codewords $\mathbf{c}$ and $\mathbf{e}$, where $*$ denotes complex conjugate. In the following, we explore the algebraic structure of the convolutional codes and present some guidelines to guarantee the full rank condition, yielding a reduced complexity code search.

Lemma 1 *Consider a rate $R = 1/2$ convolutional space-time code over GF(p), p a prime, with a generator matrix*

$$\mathbf{G}(D) = \left[\begin{array}{c} g_{1,1}D + \ldots + g_{K,1}D^K \\ g_{0,2} + \ldots + g_{K-1,2}D^{K-1} \end{array} \right]^T ,$$

where $g_{K,1} \neq 0$ and $g_{0,2} \neq 0$. Then, 1) all branches in the code trellis departing from the same state coincide in the first symbol and differ in the second symbol, and 2) all transitions arriving at the same state coincide in the second symbol and differ in the first symbol.

Proof: We shall use the fact that, for field elements u, u', and g:

$$u \cdot g = u' \cdot g \quad \Rightarrow \quad u = u', \tag{8}$$

and

$$u \cdot g \neq u' \cdot g \quad \Rightarrow \quad u \neq u'. \tag{9}$$

To prove 1), let us consider the current state as being $\sigma_s = [u_1, \ldots, u_K]^1$. When $u_0(f)$ is the input symbol, the state transition is $\sigma_s \rightarrow \sigma_f = [u_o(f), u_1, \ldots, u_{K-1}]$. Since $g_{0,1} = 0$, the coded symbol corresponding to the first antenna will be given by:

$$v^1 = \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,1} + u_K \cdot g_{K,1}, \tag{10}$$

which is a constant in GF(p) that does not depend on $u_0(f)$, but on σ_s. This shows that the first symbol of all branches departing from the same state are equal. Now for the same state transition $\sigma_s \rightarrow \sigma_f$, since $g_{K,2} = 0$, the symbol corresponding to the second antenna will be given by:

$$v^2 = u_0(f) \cdot g_{0,2} + \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,2} = u_0(f) \cdot g_{0,2} + V_2, \tag{11}$$

where V_2 is a constant in GF(p). Thus, in GF(p), v^2 will be different for different values of $u_0(f)$.

To prove 2), let us consider the final state as being $\sigma_f = [u_0, u_1, \ldots, u_{K-1}]$. Then, for the state transition $\sigma_s = [u_1, \ldots, u_{K-1}, u_K(s)] \rightarrow \sigma_f$, the symbol corresponding to the first antenna is given by:

$$v^1 = \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,1} + u_K(s) \cdot g_{K,1} = V_1 + u_K(s) \cdot g_{K,1}, \tag{12}$$

where V_1 is a constant in GF(p). So, in GF(p), v^1 will be different for different values of $u_K(s)$. For the same transition $\sigma_s \rightarrow \sigma_f$, the symbol corresponding to the second antenna will be given by:

$$v^2 = u_0 \cdot g_{0,2} + \sum_{\mu=1}^{K-1} u_\mu \cdot g_{\mu,2}, \tag{13}$$

which is a constant in GF(p) that does not depend on $u_K(s)$, only on σ_f. This completes the proof. ■

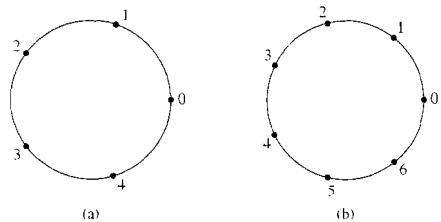[1]Note that here we drop the index t used in Fig. 2 for the sake of simplicity.

Figure 3. Constellations 5-PSK (a) and 7-PSK (b).

Theorem 1 *Consider a rate $R = 1/2$ convolutional space-time code over GF(p), p a prime. If the generator matrix $\mathbf{G}$ is in accordance with Lemma 1, then the full rank condition is guaranteed.*

Proof: Assume the generator matrix $\mathbf{G}$ is in accordance with Lemma 1, and consider an arbitrary pair of codewords $\mathbf{c}$ and $\mathbf{e}$ of length $2l$ symbols. Then, equation (7) results in:

$$\det_l \triangleq \det \left(\left[\begin{array}{cc} 0 & 0 \\ 0 & |a_1|^2 \end{array} \right] + \left[\begin{array}{cc} f & d \\ d^* & h \end{array} \right] + \left[\begin{array}{cc} |b_l|^2 & 0 \\ 0 & 0 \end{array} \right] \right), \tag{14}$$

where

$$a_i = \left(e_i^2 - c_i^2 \right), \quad b_i = \left(e_i^1 - c_i^1 \right), \quad f = \sum_{i=2}^{l-1} |b_i|^2,$$

$$h = \sum_{i=2}^{l-1} |a_i|^2, \quad \text{and} \quad d = \sum_{i=2}^{l-1} a_i b_i^*. \tag{15}$$

From Lemma 1, we have that $|a_1|^2 > 0$ and $|b_l|^2 > 0$. By Schwarz inequality,

$$\sum_i |a_i|^2 \times \sum_i |b_i|^2 \geq \left| \sum_i a_i b_i^* \right|^2, \tag{16}$$

and thus $fh \geq |d|^2$. Since $f \geq 0$ and $h \geq 0$, we must have that:

$$\det_l = \underbrace{|a_1|^2 |b_l|^2}_{>0} + \underbrace{|a_1|^2 f}_{\geq 0} + \underbrace{|b_l|^2 h}_{\geq 0} + \underbrace{fh - |d|^2}_{\geq 0} > 0, \tag{17}$$

which guarantees the full rank condition. ■

4. CODE SEARCH RESULTS

In this section, we present some examples of 2-space-time codes generated by a rate $R = 1/2$ linear convolutional encoder over GF(5) and GF(7). The restrictions derived in the previous section are applied to the convolutional encoder (i.e., $g_{K,1} \neq 0$, $g_{0,2} \neq 0$ and $g_{0,1} = g_{K,2} = 0$) to guarantee the full rank condition. The minimum determinant is then maximized. We begin with examples of unit-memory encoders.

Example 1 *Consider the 5-PSK constellation in Fig. 3(a). Following the restrictions of the previous section, the unit-memory convolutional encoder over GF(5) has generator matrix $\mathbf{G}(D) = [g_{1,1}D, g_{0,2}]$. These codes have diversity gain 2, with spectral efficiency of 2.32 b/s/Hz. In performing*
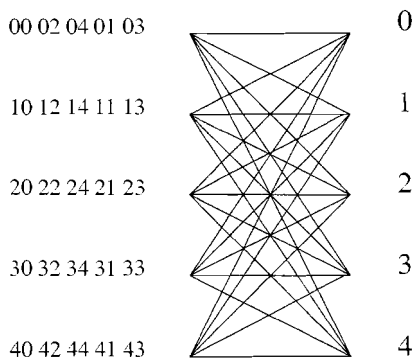
Figure 4. 2-space-time code for 5-PSK, 2.32 b/s/Hz, (convolutional encoder: GF(5), $R = 1/2$, $K = 1$).

Figure 5. 2-space-time code for 7-PSK, 2.81 b/s/Hz, (convolutional encoder: GF(7), $R = 1/2$, $K = 1$).
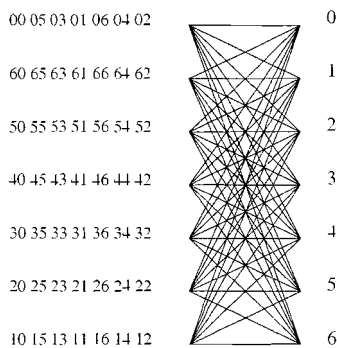

Figure 6. 2-space-time code for 5-PSK, 2.32 b/s/Hz, (convolutional encoder: GF(5), $R = 1/2$, $K = 2$).

a code search, we made $g_{1,1}$ and $g_{0,2}$ to vary over all nonzero elements in GF(5), resulting in 16 different codes. The minimum determinant of $A(\mathbf{c}, \mathbf{e})$ for each code is listed in the matrix $D(g_{1,1}, g_{0,2})$ below:

$$D(g_{1,1}, g_{0,2}) = \begin{pmatrix} 1,38 & 2,23 & 2,23 & 1,38 \\ 2,23 & 1,38 & 1,38 & 2,23 \\ 2,23 & 1,38 & 1,38 & 2,23 \\ 1,38 & 2,23 & 2,23 & 1,38 \end{pmatrix},$$

where $g_{1,1}$ is the row index and $g_{0,2}$ is the column index.

Clearly, the best code is obtained with, for instance, $g_{1,1} = 1$ and $g_{0,2} = 2$. This code has coding gain 2.23. The trellis of this code is shown in Fig.4.

In this and in the following trellises, each pair of symbols to the left of the trellis indicates the signal transmitted over the first and the second antennas, respectively, and the label to the right of the trellis indicates the state of the encoder.

Example 2 *Now consider the signal constellation 7-PSK in Fig. 3(b). Again, we consider a unit-memory encoder satisfying the conditions of Section 3. We thus have diversity gain 2, with spectral efficiency of 2.81 b/s/Hz. Similarly to Example 1, we obtained the matrix $D(g_{1,1}, g_{0,2})$ as*

$$D(g_{1,1}, g_{0,2}) = \begin{pmatrix} 0,75 & 1,35 & 1,35 & 1,35 & 1,35 & 0,75 \\ 1,35 & 0,75 & 1,35 & 1,35 & 0,75 & 1,35 \\ 1,35 & 1,35 & 0,75 & 0,75 & 1,35 & 1,35 \\ 1,35 & 1,35 & 0,75 & 0,75 & 1,35 & 1,35 \\ 1,35 & 0,75 & 1,35 & 1,35 & 0,75 & 1,35 \\ 0,75 & 1,35 & 1,35 & 1,35 & 1,35 & 0,75 \end{pmatrix}$$
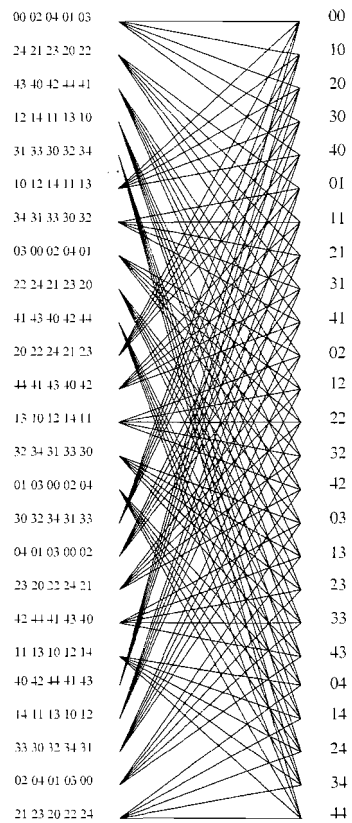
The best code is obtained with, for instance, $g_{1,1} = 6$ and $g_{0,2} = 5$. This code has coding gain 1.35. The trellis of this code is shown in Fig.5.

Example 3 *Here we extend Example 1 to the case of convolutional encoder of memory order $K = 2$ over GF(5). That is, the convolutional encoder has generator matrix $\mathbf{G}(D) = [g_{1,1}D + g_{2,1}D^2, g_{0,2} + g_{1,2}D]$. For this case, one best code is generated by $g_{1,1} = 2$, $g_{2,1} = 1$, $g_{0,2} = 2$, and $g_{1,2} = 4$ (see trellis in Fig. 6). The coding gain of this code is 4.47. The diversity gain and the spectral efficiency are the same as those in Example 1.*

Example 4 *In this example, we consider a convolutional encoder of memory order $K = 2$ over GF(7). Similarly to Example 2, the generator matrix is $\mathbf{G}(D) = [g_{1,1}D + g_{2,1}D^2, g_{0,2} + g_{1,2}D]$. The multipliers set as $g_{1,1} = 3$, $g_{2,1} = 2$, $g_{0,2} = 1$, and $g_{1,2} = 3$ produce one best code. The trellis description for this code is shown in Fig. 7. The coding gain is 3.74, while the diversity gain and the spectral efficiency are the same as those in Example 2.*

It should be noticed that the two matrices $D(g_{1,1}, g_{0,2})$ above, for the cases of unit-memory convolutional codes over GF(5) and GF(7), are symmetric. We prove next that such a symmetry holds true for any field GF(p), p a prime, and any encoder memory K. This fact can be used to limit the code
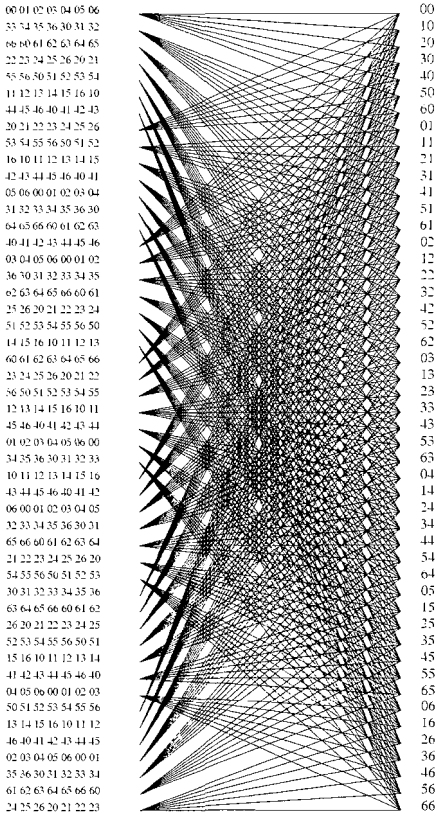
Figure 7. 2-space-time code for 7-PSK, 2.81 b/s/Hz, (convolutional encoder: GF(7), $R = 1/2$, $K = 2$).

search even further, for the cases of larger values of p and K.

Theorem 2 *Consider the space-time system model in Fig. 1. Let* $\mathbf{G}(D) = [G_1(D), G_2(D)]$ *be the generator matrix of a rate* $R = 1/2$ *convolutional code over GF(p), p a prime, with code generators* $G_1(D) = g_{0,1} + g_{1,1}D + g_{2,1}D^2 \cdots + g_{K,1}D^K$ *and* $G_2(D) = g_{0,2} + g_{1,2}D + g_{2,2}D^2 \cdots + g_{K,2}D^K$. *The space-time convolutional code generated by* $\mathbf{G}(D)$ *has the same diversity gain and coding gain as the space-time convolutional code generated by the generator matrix* $\hat{\mathbf{G}}(D) = [\hat{G}_1(D), \hat{G}_2(D)]$, *with code generators* $\hat{G}_1(D) \triangleq D^K G_1(D^{-1}) = g_{K,2} + g_{K-1,2}D + g_{K-2,2}D^2 \cdots + g_{0,2}D^K$ *and* $\hat{G}_2(D) \triangleq D^K G_2(D^{-1}) = g_{K,1} + g_{K-1,1}D + g_{K-2,1}D^2 \cdots + g_{0,1}D^K$.

Proof: Consider the convolutional encoder of Fig. 2, for the code generators $G_1(D)$ and $G_2(D)$. Let the current state be $\sigma_s = [u_1, u_2, \ldots, u_K]$, where $u_i \in$ GF(p), $i = 1, 2, \cdots, K$. The input symbol u_o causes the state transition $\sigma_s \rightarrow \sigma_f = [u_o, u_1, \ldots, u_{K-1}]$ and the coded outputs, v^1 and v^2, are:

$$v^1 = \sum_{\mu=0}^{K} u_\mu \cdot g_{\mu,1} \tag{18}$$

and

$$v^2 = \sum_{\mu=0}^{K} u_\mu \cdot g_{\mu,2}. \tag{19}$$

For the code generators $\hat{G}_1(D)$ and $\hat{G}_2(D)$, suppose that the current state is $\hat{\sigma}_s = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_K]$. In this case, the input symbol $\hat{u}_o$ causes the state transition $\hat{\sigma}_s \rightarrow \hat{\sigma}_f = [\hat{u}_o, \hat{u}_1, \ldots, \hat{u}_{K-1}]$ and the coded outputs, $\hat{v}^1$ and $\hat{v}^2$, are:

$$\hat{v}^1 = \sum_{\mu=0}^{K} \hat{u}_\mu \cdot g_{K-\mu,2} \tag{20}$$

and

$$\hat{v}^2 = \sum_{\mu=0}^{K} \hat{u}_\mu \cdot g_{K-\mu,1}. \tag{21}$$

When $u_i = \hat{u}_{K-i}$, for $i = 0, 1, 2, \ldots, K$, the coded outputs $\hat{v}^1$ and $\hat{v}^2$ are, respectively, v^2 and v^1, and the corresponding state transition is $\hat{\sigma}_s = [u_{K-1}, \ldots, u_1, u_0] \rightarrow \hat{\sigma}_f = [u_K, \ldots, u_2, u_1]$. The code generated by $\hat{\mathbf{G}}(D)$ is called the reciprocal code of the code generated by $\mathbf{G}(D)$. It follows that for every pair of space-time codewords

$$\mathbf{c} = c_1^1 c_1^2 \cdots c_1^n c_2^1 c_2^2 \cdots c_2^n \cdots c_l^1 c_l^2 \cdots c_l^n,$$

and

$$\mathbf{e} = e_1^1 e_1^2 \cdots e_1^n e_2^1 e_2^2 \cdots e_2^n \cdots e_l^1 e_l^2 \cdots e_l^n$$

in the original code, there is a pair of space-time codewords

$$\hat{\mathbf{c}} = c_l^n \cdots c_l^2 c_l^1 \cdots c_2^n \cdots c_2^2 c_2^1 c_1^n \cdots c_1^2 c_1^1,$$

and

$$\hat{\mathbf{e}} = e_l^n \cdots e_l^2 e_l^1 \cdots e_2^n \cdots e_2^2 e_2^1 e_1^n \cdots e_1^2 e_1^1,$$

in the reciprocal code. Since both pairs of codewords yield matrices $B(\mathbf{c}, \mathbf{e})$ with the same rank and the same determinant, the two space-time codes must have the same diversity gain and coding gain.

■

5. SIMULATION RESULTS

In this section, we present simulation results for the four 2-space-time codes designed in the last section. We consider the cases of $m = 1, 2, 3$ receive antennas and frame length $l = 130$ symbols.

Figure 8 shows the frame error rate (FER) versus the signal-to-noise ratio (SNR), in decibels, for the codes constructed over $GF(5)$, with $K = 1, 2$. As we can predict from (6), for the same number of receive antennas, the code with memory order $K = 2$, and coding gain 4.47, has a better performance than the code with unitary memory, and coding gain 2.23. With the increase in the number of receiving antennas, the performance is ameliorated for both cases, $K = 1, 2$, while the relative advantage of the code with memory order $K = 2$ is magnified. Again, this behavior can be induced from (6).

In Figure 9 we have the same comparisons as in Figure 8, but this case for the codes constructed over $GF(7)$. As in Figure 8, the results are in accordance with (6), what validates our code construction method and search results.

6. FINAL COMMENTS

In this paper, we considered space-time convolutional codes over GF(p) for the quasi-static, flat Rayleigh channel.
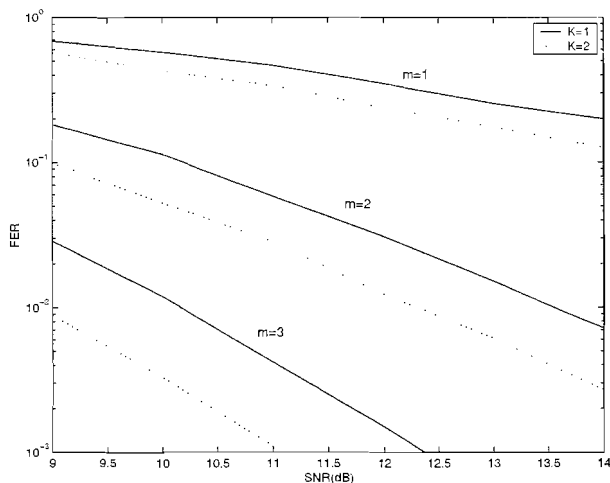
Figure 8. Frame error rate (FER) versus signal-to-noise ratio (SNR) for the 2-space-time codes for 5-PSK with $K = 1, 2$ and $m = 1, 2, 3$ receive antennas.

Figure 9. Frame error rate (FER) versus signal-to-noise ratio (SNR) for the 2-space-time codes for 7-PSK with $K = 1, 2$ and $m = 1, 2, 3$ receive antennas.

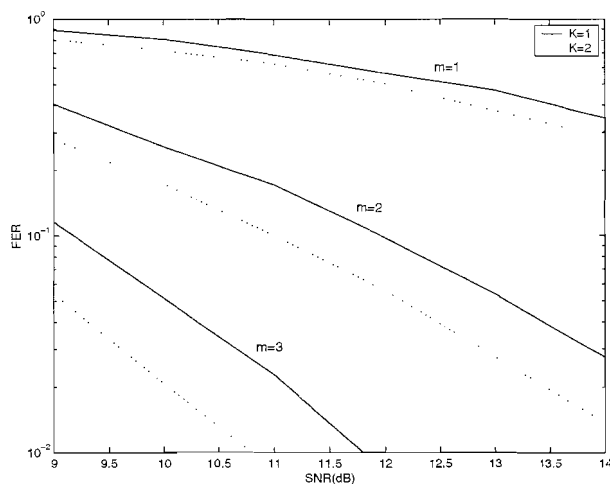The codes were designed to provide the best performance according to the rank and the determinant criteria derived by Tarokh *et. al.*. Simple conditions on the generator matrices of a rate $1/2$ convolutional code were given that guarantee maximum diversity advantage for 2 transmit antennas.

New space-time convolutional codes for 2 transmit antennas for the 5-PSK and 7-PSK constellations were presented. Spectral efficiencies of 2.32 and 2.81 b/s/Hz, respectively, were achieved, providing more options than the usual 2 and 3 b/s/Hz.

Simulation results in the form of frame error rate versus signal-to-noise ratio were reported for these codes for $m = 1, 2, 3$ receive antennas. The corresponding coding and diversity advantages were verified, in agreement with the theoretical results presented in Section 3.

The space-time codes presented in this paper could be found with a reduced computer search, due to the linearity of the convolutional codes, the restrictions derived in Section 3,

and the symmetry of the matrix D in Section 4. Specifically, an exhaustive code search for the cases of the examples in Section 4 would need to check a huge number ($O(p^{(p^K \, 2p)})$) of codes. By considering the convolutional codes which satisfy the conditions of Theorem 1, one would need to check only $O(p^{2K})$ codes. The number of codes that have to be checked can be reduced further if one takes the symmetry of Theorem 2 into account. For instance, in Example 1 only 10 candidates were checked and, for the more complex case, in Example 4, there were 735 candidates. These advantages become more significant as the memory order increases, allowing good codes to be obtained with relatively low computer effort.

The fact that p is not a power of 2 may pose as a problem if transmission of binary data is required. We tackle this problem by associating a different word of n_p p-ary symbols to each word of n_2 bits, where $p^{n_p} > 2^{n_2}$. The spectral efficiency is $n_2/n_p < \log_2(p)$ b/s/Hz, that is, there is a percentual loss of

$$\left[1 - \left(\frac{n_2}{n_p \log_2(p)} \right) \right] \times 100.$$

Clearly this loss is minimized by maximizing the relation $\frac{n_2}{n_p \log_2(p)}$, where $p^{n_p} > 2^{n_2}$. For instance, consider that, in the case of the 5-PSK constellation, each word of $n_2 = 9$ bits is associated to a different word of $n_p = 4$ symbols. Then, the spectral efficiency will be 2.25 b/s/Hz, producing a loss of only 3%.

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